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Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1996

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MECANICĂ TEORETICĂ
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1996

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Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

S U M A R

W. B. VASANTHA KANDASAMY (India), Inele tri-operaționale polinomiale (engl., rez. rom.)	1
GH. COSTOVICI, Unele grupuri pe unele produse carteziene (engl., rez. rom.)	5
C. CĂLIN, CR-Subvarietăți de contact normale ale unei varietăți trans-Sasaki (engl., rez. rom.)	9
M. TURINICI, Rezultate de comparație diferențială peste mulțimi excepționale (engl., rez. rom.)	17
NICOLETA NEGOESCU, Puncte fixe comune pentru multifuncționale care satisfac inegalități raționale (franc., rez. rom.)	29
ADRIAN CORDUNEANU și GH. MOROȘANU, Asupra sistemului $\dot{x}=A(t)x$ cu $A(t)$ simetrică sau antisimetrică (engl., rez. rom.)	35
I. ONCIULESCU, Baze Jordan și ecuații diferențiale liniare de ordin superior, cu coeficienți constanți (engl., rez. rom.)	41
MARIA BECEANU și ARIADNA-LUCIA PLETEA, Metoda elementului finit pentru o problemă singulară cu valori la limită (engl., rez. rom.)	49
VIOREL GHIUȘ, Tipuri de circulații admisibile într-un graf scară neorientat (III) (franc., rez. rom.)	55

JESÚS LÓPEZ-FIDALGO și GUSTAVO SANTOS-GARCÍA (Spania), Modele Markov ascunse în recunoașterea automată a vorbirii. O aplicație independentă de vorbitor la un vocabular de cuvinte spaniole (engl., rez. rom.)	59
LIVIU BĂDELIȚĂ și VIOREL STANCU, Asupra comportării neliniare a unui oscilator de tip Duffing (engl., rez. rom.)	71
A. K. GHOSH și A. R. KHAN (India), Asupra problemei lui Ekman pentru fluide vâscoelastice liniare (engl., rez. rom.)	83
MARICEL AGOP, N. REZLESCU, CĂLIN GH. BUZEA și CRISTINA BUZEA, Soluția "kink" a ecuației Ginzburg-Landau. Analiză topologică (engl., rez. rom.)	93
DORIAN ȚĂȚOMIR și ADRIAN SACHELARIE, Fotoproducerea gravitonilor în câmpuri electromagnetice exterioare (engl., rez. rom.)	99
D. CORDUNEANU și EMIL LUCA, Model Fuzzy privind optimizarea procesului de precarbonizare a fibrelor PAN termostabilizate (engl., rez. rom.)	105
MAGDA GHERGHEL, Procese fotovoltaiice de interfață la heterostructuri tip multistrat pe bază de compuși semiconductori II-VI și pigmenți fotosintetici (engl., rez. rom.)	123
M. L. CRAUS și EMIL LUCA, Asupra transformărilor structurale induse în unele aliaje Fe-Cr-Ni de implantarea cu $^{32}\text{S}(+5)$ (engl., rez. rom.)	131
SERGIU ISTRATE, Proprietăți magnetice ale feritelor Mn-Zn impurificate cu TiO_2 (engl., rez. rom.)	139
RECENZII	145

Section I

MATHEMATICS. THEORETICAL MECHANICS. PHYSICS

C O N T E N T S

W. B. VASANTHA KANDASAMY (India), Polynomial Tri-Operational Rings (English, Romanian summary)	1
GH. COSTOVICI, Some Groups on Some Cartesian Products (English, Romanian summary)	5
C. CĂLIN, Normal Contact CR-Submanifolds of a Trans-Sasakian Manifold (English, Romanian summary)	9
M. TURINICI, Differential Comparison Results over Exceptional Sets (English, Romanian summary)	17
NICOLETA NEGOESCU, Points fixes communs pour des applications multi- voques qui satisfont aux inégalités rationnelles (French, Romanian summary) .	29
ADRIAN CORDUNEANU and GH. MOROȘANU, On the System $\dot{x}=A(t)x$ with $A(t)$ Symmetric or Antisymmetric (English, Romanian summary)	35
I. ONCIULESCU, Jordan Bases and Linear Differential Equations of Higher Order with Constant Coefficients (English, Romanian summary)	41
MARIA BECEANU and ARIADNA-LUCIA PLETEA, The Finite Element Methods for a Singular Boundary Value Problem (English, Romanian su- mmary)	49

VIOREL GHIUȘ, Les types de circulations admissibles dans un graphe escalier non orienté (III) (French, Romanian summary)	55
JESÚS LÓPEZ-FIDALGO and GUSTAVO SANTOS-GARCÍA (Spain), Hidden Markov Models in Automatic Speech Recognition. A Speaker-Independent Application to a Vocabulary of Spanish Words (English, Romanian summary)	59
LIVIU BĂDELIȚĂ and VIOREL STANCU, About the Non-linear Behaviour of Some Duffing Type Oscillating Systems (English, Romanian summary)	71
A. K. GHOSH and A. R. KHAN (India), On Ekman Problem for Linear Visco-elastic Fluids (English, Romanian summary)	83
MARICEL AGOP, N. REZLESCU, CĂLIN GH. BUZEA and CRISTINA BUZEA, The Kink Solution of the Ginzburg-Landau Equation. Topological Analysis (English, Romanian summary)	93
DORIAN ȚĂȚOMIR and ADRIAN SACHELARIE, The Graviton Photoproduction in External Electromagnetic Fields (English, Romanian summary)	99
D. CORDUNEANU and EMIL LUCA, Fuzzy Model Regarding the Optimization of the Process of Precarbonization of Thermostabilized PAN Fibres (English, Romanian summary)	106
MAGDA GHERGHEL, The Photoelectrical Interface Processes in Multilayer-Type Heterostructures Based on II-VI Semiconductors Compounds and Photo-synthetic Pigment (English, Romanian summary)	123
M. L. CRAUS and EMIL LUCA, On the Structure Induced Transformations by $^{32}\text{S}(+5)$ Implantation at Some Fe-Cr-Ni Alloys (English, Romanian summary)	131
SERGIU ISTRATE, Magnetic Properties of the Mn-Zn Ferrites Impurified with TiO_2 (English, Romanian summary)	139
BOOK REVIEWS	145

POLYNOMIAL TRI-OPERATIONAL RINGS

BY

W. B. VASANTHA KANDASAMY

Abstract. The notion of polynomial tri-operational ring are introduced and some interesting properties about these rings are obtained.

Key words: near-ring, polynomial ring, polynomial tri-operational ring.

In this paper we introduce the notion of *polynomial tri-operational ring* which are just like polynomial rings. Here we study and obtain some interesting properties about these rings. G. Pilz in [1] has defined composition ring or tri-operational ring as follows:

Definition (G. Pilz [1]). A set R together with three binary operations "+", "·", "◦" is called a composition ring (tri-operational algebra) if $(R, +, ·)$ is a ring, $(R, +, ◦)$ a near-ring and if for all $\gamma, \gamma', \gamma''$ in R : $(\gamma, \gamma') \circ \gamma'' = (\gamma \circ \gamma') \cdot (\gamma' \circ \gamma'')$.

Example 1. Let $T = \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$ be the set of positive integers, negative integers with zero. T under usual addition and multiplication is a ring. T with the same usual addition and the operation "◦" $a \circ b = a$ for all $a, b \in T$ is a near-ring. Obviously $(a \cdot b) \circ c = (a \circ c) \cdot (b \circ c)$ for all $a, b, c \in T$. T is a tri-operational ring.

Example 2. Let $T = \mathbb{Q}$ the set of positive and negative rationals together with zero. With the same operations "+", "·", "◦" defined in Example 1, $\mathbb{Q}$ is a tri-operational ring.

We say T is commutative if T as a ring is commutative.

Definition 1. Let T be a tri-operational ring. By the tri-operational ring of polynomials in the indeterminate x , written as $T[x]$, we mean the set of all symbols $a_0 + a_1x + \dots + a_nx^n$, where n can be any non negative integer and where the coefficients $a_1, a_2, \dots, a_n$ are all in T .

In order to make a ring out of $T[x]$ we must be able to recognize when two ele-

ments in it are equal, we must be able to add and multiply elements of $T[x]$, so that the axioms defining a ring hold true for $T[x]$.

Definition 2. If $p(x) = a_0 + a_1x + \dots + a_mx^m$ and $q(x) = b_0 + b_1x + \dots + b_nx^n$ are both in $T[x]$ then $p(x) = q(x)$ if and only if for every integer $i \geq 0$, $a_i = b_i$. Thus two polynomials are declared equal if and only if their corresponding coefficients are equal.

Definition 3. If $p(x) = a_0 + a_1x + \dots + a_mx^m$ and $q(x) = b_0 + b_1x + \dots + b_nx^n$ are both in $T[x]$, then $p(x) + q(x) = c_0 + c_1x + \dots + c_kx^k$ where for each i , $c_i = a_i + b_i$.

Definition 4. If $p(x) = a_0 + a_1x + \dots + a_mx^m$ and $q(x) = b_0 + b_1x + \dots + b_nx^n$, then $p(x) \cdot q(x) = c_0 + c_1x + \dots + c_kx^k$ where $c_i = a_ib_0 + a_{i-1}b_1 + \dots + a_0b_i$.

Definition 5. If $p(x) = a_0 + a_1x + \dots + a_mx^m$ and $q(x) = b_0 + b_1x + \dots + b_nx^n$, then $p(x) \circ q(x) = p(x)$. Thus $(T[x], +, \cdot, \circ)$ is a tri-operational algebra, called the polynomial tri-operational algebra.

Now if $f(x) = a_0 + a_1x + \dots + a_nx^n \neq 0$ and $a_n \neq 0$, then the degree of $f(x)$, written as $\deg f(x)$, is n .

If $f(x), g(x)$ are two non-zero elements of $T[x]$, then $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$, provided $(T, +, \cdot)$ as a ring has no divisors of zero.

If $f(x), g(x)$ are two non-zero elements of $T[x]$, then $\deg(f(x) \circ g(x)) = \deg f(x)$.

Corollary. $T[x]$ is a tri-operational algebra.

Definition 6. Let $T[x]$ be a tri-operational polynomial ring. A non empty subset I_R of $T[x]$ is a right ring ideal " $\cdot$ " if I_R is a ideal of the ring $(T[x], +, \cdot)$.

Definition 7. Let $T[x]$ be a tri-operational polynomial ring. A non empty subset N_R of $T[x]$ is a right near-ring ideal of $T[x]$ if N_R is a ideal of the near-ring $(T[x], +, \circ)$.

Definition 8. Let $T[x]$ be a tri-operational polynomial ring. A non empty subset J_R of $T[x]$ is a right tri-operational ideal of $T[x]$ if

- (i) J_R is a right ring ideal,
- (ii) J_R is a right near-ring ideal of $T[x]$.

Theorem 9. Let $T[x]$ be a tri-operational polynomial ring. Every polynomial $p(x)$ of degree greater than or equal to one generates a right near-ring tri-operational ideal.

Proof. Obvious.

Theorem 10. Let $T[x]$ be a tri-operational polynomial ring. Every polynomial $p(x)$ of $T[x]$ in general need not be a right ring ideal of $T[x]$.

Proof. Obvious.

Theorem 11. *Let $T[x]$ be a tri-operational polynomial ring. Every polynomial $p(x)$ of $T[x]$ is an idempotent under " $\circ$ ".*

Proof. Obvious, as $p(x) \circ p(x) = p(x)$ for every $p(x) \in T[x]$.

Theorem 12. *Let $T[x]$ be a tri-operational polynomial ring. Every polynomial $p(x)$ of $T[x]$ is reducible under " $\circ$ ".*

Proof. Obvious, for every $p(x) \in T[x]$ we can write $p(x) = p(x) \cdot \gamma(x)$ for every $\gamma(x) \in T[x]$. Thus a polynomial irreducible under " $\cdot$ " is reducible under " $\circ$ ".

Theorem 13. *Let $T[x]$ be a polynomial ring. $T[x]$ is a principal ideal ring, if $(T, +, \cdot)$ is a field.*

Proof. $T[x]$ is a principal ideal ring, if $(T, +, \cdot)$ is a field, since under " $\circ$ " every polynomial generates an ideal. The theorem is true.

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Indian Institute of Technology,
Department of Mathematics,
Madras, India

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INELE TRI-OperaȚIONALE POLINOMIALE

(Rezumat)

Se introduce noțiunea de inel tri-operational polinomial și se stabilesc o serie de proprietăți interesante ale lor.

SOME GROUPS ON SOME CARTESIAN PRODUCTS

BY

GH. COSTOVICI

Abstract. Some groups on some cartesian products are introduced and then studied the connections between them and also their ordering.

Key words: group, semi-group, cartesian product.

In this Note we define some groups on some cartesian products and study the connections between them and the ordering of them. All these are presented in the following

Theorem. Let A and B be groups, $F = \{f: B \rightarrow A\}$ and $G = B_x F$. Then we have I, II_i, $i = \overline{1, 4}$, III_j, $j = \overline{1, 3}$:

I $(G, \circ_i)$ is a group, $i = \overline{1, 4}$, where

$$(b_1, f_1) \circ_1 (b_2, f_2) = (b_1 b_2, x \rightarrow f_1(x b_2^{-1}) f_2(x)),$$

$$(b_1, f_1) \circ_2 (b_2, f_2) = (b_1 b_2, x \rightarrow f_1(b_2 x) f_2(x)),$$

$$(b_1, f_1) \circ_3 (b_2, f_2) = (b_1 b_2, x \rightarrow f_1(x) f_2(b_1^{-1} x)),$$

$$(b_1, f_1) \circ_4 (b_2, f_2) = (b_1 b_2, x \rightarrow f_1(x) f_2(x b_1));$$

II₁ if B is commutative then $\varphi: (G, \circ_1) \rightarrow (G, \circ_2)$ with $\varphi(b, f) = (b^{-1}, f)$ is isomorphism;

II₂ if B is commutative then $\varphi: (G, \circ_3) \rightarrow (G, \circ_4)$ with $\varphi(b, f) = (b^{-1}, f)$ is isomorphism;

II₃ if A and B are commutative then $1_G : (G, \circ_1) \rightarrow (G, \circ_3)$ with $1_G(b, f) = (b, f)$ is anti-isomorphism;

II₄ if A and B are commutative then $1_G : (G, \circ_2) \rightarrow (G, \circ_4)$ with $1_G(b, f) = (b, f)$ is anti-isomorphism;

III₁ if A and B are right $p.$ o. groups then $(G, \circ_i)$, $i = \overline{1, 4}$, is right $p.$ o. group by $(b_1, f_1) \leq (b_2, f_2) \Leftrightarrow b_1 < b_2$ or $(b_1 = b_2 \text{ and } f_1 \leq f_2; f_1 \leq f_2 \Leftrightarrow \forall b \in B, f_1(b) \leq f_2(b))$;

III₂ if A and B are left $p.$ o. groups then $(G, \circ_i)$, $i = \overline{1, 4}$, is left $p.$ o. group by the relation of III₁;

III₃ if A and B are $p.$ o. groups then $(G, \circ_i)$, $i = \overline{1, 4}$, is $p.$ o. group by the relation of III₁.

Before to outline the proof of the Theorem we mention that we have been inspired from [1], only that there the operation

$$(b_1, f_1) \circ (b_2, f_2) = (b_1 b_2, x \rightarrow f_1(b_2^{-1} x) f_2(x))$$

is not associative. Perhaps there is a printing error.

P r o o f of the Theorem.

$$\begin{aligned} [(b_1, f_1) \circ_1 (b_2, f_2)] \circ_1 (b_3, f_3) &= (b_1 b_2, x \rightarrow f_1(x b_2^{-1}) f_2(x)) \circ_1 (b_3, f_3) = \\ &= (b_1 b_2 b_3, x \rightarrow f_1(x b_3^{-1} b_2^{-1}) f_2(x b_3^{-1}) f_3(x)), \end{aligned}$$

$$\begin{aligned} (b_1, f_1) \circ_1 [(b_2, f_2) \circ_1 (b_3, f_3)] &= (b_1, f_1) \circ_1 (b_2 b_3, x \rightarrow f_2(x b_3^{-1}) f_3(x)) = \\ &= (b_1 b_2 b_3, x \rightarrow f_1(x b_3^{-1} b_2^{-1}) f_2(x b_3^{-1}) f_3(x)). \end{aligned}$$

So the associativity of $\circ_1$ is fulfilled. The checking of the associativity of $\circ_i$, $i = \overline{2, 4}$, can be made similarly.

If $i : B \rightarrow A$ with $i(b) = 1 \forall b \in B$, then $(b, f) \circ_1 (1, i) = (b, x \rightarrow f(x) i(x)) = (b, x \rightarrow f(x)) = (b, f)$. If $(b, f) \in G$, $\exists (b^{-1}, x \rightarrow (f(xb))^{-1}) \in G$ so that $(b, f) \circ_1 (b^{-1}, x \rightarrow (f(xb))^{-1}) = (1, x \rightarrow f(xb)(f(xb))^{-1}) = (1, x \rightarrow 1) = (1, i)$. So $(G, \circ_1)$ is a group.

We mention, also, that $(1, i)$ is the neutral element for $(G, \circ_i)$, $i = \overline{2, 4}$. The inverse of (b, f) with respect to $\circ_2$ is $(b^{-1}, x \rightarrow (f(b^{-1}x))^{-1})$, with respect to $\circ_3$ is $(b^{-1}, x \rightarrow (f(bx))^{-1})$ and with respect to $\circ_4$ is $(b^{-1}, x \rightarrow (f(xb^{-1}))^{-1})$. We verify the ordering of $(G, \circ_1)$ from III₃ and omit the rest of the proof:

$$(b_1, f_1) \leq (b_2, f_2) \Leftrightarrow b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } f_1 \leq f_2),$$

$$(b_1, f_1) \circ_1 (b, f) = (b_1 b, x \rightarrow f_1(xb^{-1}f(x))),$$

$$(b_2, f_2) \circ_1 (b, f) = (b_2 b, x \rightarrow f_2(xb^{-1}f(x)));$$

$$b_1 < b_2 \Rightarrow b_1 b < b_2 b \Rightarrow (b_1, f_1) \circ_1 (b, f) \leq (b_2, f_2) \circ_1 (b, f);$$

$$b_1 = b_2 \text{ and } f_1 \leq f_2 \Rightarrow b_1 b = b_2 b \text{ and } f_1(xb^{-1}f(x)) \leq f_2(xb^{-1}f(x)) \Rightarrow$$

$$\Rightarrow (b_1, f_1) \circ_1 (b, f) \leq (b_2, f_2) \circ_1 (b, f);$$

$$(b, f) \circ_1 (b_1, f_1) = (bb_1, x \rightarrow f(xb_1^{-1})f_1(x)),$$

$$(b, f) \circ_1 (b_2, f_2) = (bb_2, x \rightarrow f(xb_2^{-1})f_2(x));$$

$$b_1 < b_2 \Rightarrow bb_1 < bb_2 \Rightarrow (b, f) \circ_1 (b_1, f_1) \leq (b, f) \circ_1 (b_2, f_2);$$

$$b_1 = b_2 \text{ and } f_1 \leq f_2 \Rightarrow bb_1 = bb_2 \text{ and } f(xb_1^{-1})f_1(x) \leq f(xb_2^{-1})f_2(x) \Rightarrow (b, f) \circ_1 (b_1, f_1) \leq (b, f) \circ_1 (b_2, f_2).$$

Remark. There are similar Theorems for the case when A and B are p. o. monoids and $F_1 = \{f: B \rightarrow A \text{ increasing}\}$ and for the case when A and B are p. o. semigroups and $F_2 = \{f: B \rightarrow A \text{ decreasing}\}$.

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Department of Mathematics

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UNELE GRUPURI PE UNELE PRODUSE CARTEZIENE

(Rezumat)

Pe anumite produse carteziane se introduc mai multe structuri de grup precum și alte structuri și se studiază problema ordonării lor.

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NORMAL CONTACT CR-SUBMANIFOLDS OF A TRANS-SASAKIAN MANIFOLD

BY

C. CĂLIN

Abstract. A. Bejancu has introduced the concept of CR-submanifold of a Kählerian manifold. In the present paper one defines normal CR-submanifolds of a trans-Sasakian manifold and fundamental results on their geometry are established.

Key words: manifold, Sasakian manifold, submanifold, CR-submanifold.

0. Introduction. The concept of CR-submanifold of a Kählerian manifold has been defined by A. Bejancu [3] and is studied by many authors [10], [13]. Later A. Bejancu and N. Papaghiu [5] introduced and studied the notion of semi-invariant submanifold of a Sasakian manifold. These submanifolds are closely related to the CR-submanifolds in a Kählerian manifold. However, the existence of the structure vector field implies some important changes. Extensions of CR-submanifolds of a Sasakian manifold have also been studied by M. Kon and K. Yano [13].

The purpose of the present paper is to define what we call *normal CR-submanifolds of a trans-Sasakian manifold* and to obtain fundamental results on their geometry. In the first section we recall some results and formulae for later use. In section two, we prove some important properties of normal contact CR-submanifolds of a trans-Sasakian manifold.

1. Preliminaries. Let $\tilde{M}$ be a real $2n+1$ -dimensional differentiable manifold and f , ξ and η be a tensor field of type $(1,1)$, a vector field, a 1-form, respectively, and a Riemannian metric g , on $\tilde{M}$ satisfying

$$(1.1) \quad f^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad f(\xi) = 0, \quad \eta \circ f = 0,$$

$$(1.2) \quad g(X, Y) = g(fX, fY) + \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\tilde{M}),$$

where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$

According to D. E. Blair [6] we say that $\tilde{M}$ is an almost contact metric manifold and (f, ξ, η, g) is the almost contact metric structure on $\tilde{M}$. Throughout the

paper, all manifolds and maps are differentiable of class C^∞ . We denote by $\mathcal{F}(\tilde{M})$ the module of the differentiable function on $\tilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\tilde{M})$ -module of sections of a vector bundle E over $\tilde{M}$. We use the same notations for any other manifold involved in the study.

Next, we define the fundamental 2-form Ω of tensor f on $\tilde{M}$, by

$$\Omega(X, Y) = g(X, fY), \quad \forall X, Y \in \Gamma(TM).$$

The Nijenhuis tensor N_f of f is defined by

$$N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY], \quad \forall X, Y \in \Gamma(TM).$$

We say that the almost contact structure (f, ξ, η) is *normal* if the following condition is satisfied

$$N_f(X, Y) + 2d\eta(X, Y)\xi = 0, \quad \forall X, Y \in \Gamma(TM).$$

Finally, we say that $\tilde{M}$ is a *trans-Sasakian manifold* if it is a normal almost contact metric manifold $\tilde{M}$ which satisfies

$$(\tilde{\nabla}_X f)Y = \alpha[g(X, Y)\xi - \eta(X)Y] + \beta[g(fX, Y)\xi - \eta(Y)fX],$$

(1.3)

$$\forall X, Y \in \Gamma(TM),$$

where $\tilde{\nabla}$ is the Levi-Civita connection with respect to the metric tensor g , and α and β are function on $\tilde{M}$.

Now, let M be a m -dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of M be tangent to M . We denote by TM and $TM^\perp$ the tangent bundle to M and the normal bundle to M , respectively. We also denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M .

The submanifold M is called a *contact CR-submanifold* if it is endowed with the pair of distributions $(D, D^\perp)$ satisfying the following conditions (cf. [1]):

- i) $TM = D \oplus D^\perp \oplus \{\xi\}$ and $D, D^\perp$ and $\{\xi\}$ are orthogonal on each other;
- ii) distribution D is invariant under f , i.e., $fD_x \subseteq D_x, \forall x \in M$;
- iii) distribution $D^\perp$ is anti-invariant under f : $fD_x^\perp \subseteq T_x M^\perp, \forall x \in M$.

Throughout the paper we denote by $2p$ (resp. q) the dimension of D_x (resp. $D_x^\perp$). Thus, if $p=0$ (resp. $q=0$) then CR-submanifold is an anti-invariant submanifold tangent to ξ (resp. an invariant submanifold). An anti-holomorphic submanifold is a CR-submanifold which satisfies $\dim T_x M^\perp = q$ for each $x \in M$ (see [3]). The CR-submanifold M is called a *proper CR-submanifold* if it is neither an invariant submanifold nor an anti-invariant submanifold.

The projection morphisms from TM to D and $D^\perp$ are denoted by P and Q , respectively, and we have

$$(1.4) \quad X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

The vector fields fX, fN are decomposed into tangent parts, tX, BN , and normal parts $\omega X, CN$, as follows:

$$(1.5) \quad fX = tX + \omega X, \quad \forall X \in \Gamma(TM),$$

$$(1.6) \quad fN = BN + CN, \quad \forall N \in \Gamma(TM^\perp).$$

It is easy to see that t and C are f -structures in the sense of K. Yano [13] on TM

and $TM^\perp$, respectively.

Next, let us recall (from general theory of Riemannian submanifolds) the Gauss and Weingarten formulae

$$(1.7) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(1.8) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad \forall X \in \Gamma(TM), N \in \Gamma(TM^\perp),$$

where h is the second fundamental form of M , A_N is the shape operator with respect to the normal section N , ∇ and $\nabla^\perp$ are the connections induced by $\tilde{\nabla}$ on TM and $TM^\perp$, respectively. We also have

$$(1.9) \quad g(A_N X, Y) = g(h(X, Y), N), \quad \forall X, Y \in \Gamma(TM) \text{ and } N \in \Gamma(TM^\perp).$$

Form (1.3) we deduce

$$(1.10) \quad \tilde{\nabla}_X \xi = -\alpha fX + \beta X - \beta \eta(X)\xi, \quad \forall X, X \in \Gamma(\tilde{TM}),$$

and

$$(1.11) \quad h(X, \xi) = -\alpha \omega X, \quad A_{\omega X} \xi = -\alpha QX, \quad \forall X \in \Gamma(TM).$$

Now, we define the Nijenhuis tensor of t

$$N_t(X, Y) = [tX, tY] + t^2[X, Y] - t[X, tY] - t[tX, Y], \quad \forall X, Y \in \Gamma(TM).$$

We say that a contact CR-submanifold M of a trans-Sasakian manifold is *mixed geodesic* if $h(X, Z) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$. It is easy to see that, for any trans-Sasakian manifold, we have

L e m m a 1.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then we have*

$$(1.12) \quad (\nabla_X t)Y = A_{\omega Y}X + Bh(X, Y) + \alpha[g(X, Y)\xi - \eta(Y)X] + \\ + \beta[g(tX, Y)\xi - \eta(Y)X], \quad \forall X, Y \in \Gamma(TM),$$

and

$$(1.13) \quad (\nabla_X \omega)Y = Ch(X, Y) - h(X, tY) - \beta \eta(Y)\omega X, \quad \forall X, Y \in \Gamma(TM),$$

where $(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y$ and $(\nabla_X \omega)Y = \nabla_X^\perp \omega Y - \omega \nabla_X Y$.

L e m m a 1.2. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then we have*

$$A_{fZ}W = A_{fW}Z, \quad \forall Z, W \in \Gamma(D^\perp).$$

2. Normal Contact CR-submanifold of a Trans-Sasakian Manifold. The purpose of this section is to study the fundamental properties of a normal contact CR-submanifold. The triplet (t, ω, η) is called a *contact CR-structure* on M . First, by using (1.1) and (1.11), we infer

$$(2.1) \quad g(X, Y) = g(tX, tY) + g(\omega X, \omega Y) + \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(\tilde{TM}).$$

By using (2.1) we deduce

L e m m a 2.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then we have*

$$(2.2) \quad \Omega(tX, tY) = \Omega(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

By means of a contact CR-structure (t, ω, η) , we define the tensor field S on

M , by

$$(2.3) \quad S(X, Y) = N_t(X, Y) - 2Bd\omega(X, Y) + 2\Omega(X, Y)\xi + \\ + \beta\eta(X)QY - \beta\eta(Y)QX, \quad \forall X, Y \in \Gamma(TM).$$

The tensor field S is called the *torsion tensor* of a contact CR-structure. We say that the contact CR-submanifold M is a *normal contact CR-submanifold* if the tensor S vanishes identically on M . Using (1.11) and taking into account that ∇ is a torsion-free linear connection, (2.3) becomes

$$(2.4) \quad S(X, Y) = (\nabla_{tX}t)Y - (\nabla_{tY}t)X + t\{(\nabla_Yt)X - (\nabla_Xt)Y\} - \\ - 2Bd\omega(X, Y)\xi + \beta\eta(X)QY - \beta\eta(Y)QX, \quad \forall X, Y \in \Gamma(TM).$$

By straightforward calculation and using (1.2), (1.3) and (2.4) we infer.

L e m m a 2.2. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then we have*

$$S(X, Y) = (A_{\omega Y} \odot t - t \odot A_{\omega Y})X - (A_{\omega X} \odot t - t \odot A_{\omega X})Y, \quad \forall X, Y \in \Gamma(TM).$$

T h e o r e m . 2.1. *The contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is normal if and only if*

$$(2.5) \quad A_{\omega Y}tX = tA_{\omega Y}X, \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp).$$

P r o o f. It is easy to see that if $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D^\perp)$ or $X = \xi$ and $Y \in \Gamma(D)$, we obtain $S(X, Y) = 0$ and $S(\xi, Y) = 0, \forall Y \in \Gamma(D^\perp)$. Next, for $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, we deduce

$$(2.6) \quad S(X, Y) = A_{\omega Y}tX - tA_{\omega Y}X.$$

Finally, our assertion follows from (2.6).

R e m a r k 2.1. From the proof of the Theorem 2.2 we deduce that $S(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$ if and only if $S(X, Y) = 0$ for $\forall X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

C o r o l l a r y 2.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is normal contact CR-submanifold if*

$$(2.7) \quad h(X, tY) + h(tX, Y) \in \Gamma(\mu), \quad \forall X, Y \in \Gamma(D),$$

$$(2.8) \quad h(tX, W) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), W \in \Gamma(D^\perp).$$

P r o o f. Let $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. By using (1.2), (1.9) and (1.10) we infer

$$(2.9) \quad g(A_{\omega Z}tX - tA_{\omega Z}X, Y) = g(h(tX, Y) + h(X, tY), \omega Z),$$

$$(2.10) \quad g(A_{\omega Z}tX - tA_{\omega Z}X, W) = g(h(tX, W), \omega Z),$$

$$(2.11) \quad g(A_{\omega Z}tX - tA_{\omega Z}X, \xi) = g(\tilde{\nabla}_{tX}\xi, fZ) = 0.$$

Using Theorem 2.1, our assertion follows from (2.9)-(2.11).

From Corollary 2.1 we deduce

C o r o l l a r y 2.2. *Let M be an anti-holomorphic submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is normal contact CR-submanifold if and only if*

$$h(X, tY) + h(tX, Y) = 0, \quad \forall X, Y \in \Gamma(D)$$

and

$$h(tX, W) = 0, \quad \forall X \in \Gamma(D), W \in \Gamma(D^\perp).$$

By using Corollary 2.2 we deduce

Corollary 2.3. *Each normal contact anti-holomorphic CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is mixed geodesic.*

Corollary 2.4. *Let M be an anti-holomorphic CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. If M is a normal contact CR-submanifold and the distribution $D \oplus \{\xi\}$ is integrable, then any leaf of $D \oplus \{\xi\}$ is totally geodesic.*

Proof. Because $D \oplus \{\xi\}$ is integrable, we have $h(tX, Y) = h(X, tY)$, for any $X, Y \in \Gamma(D \oplus \{\xi\})$, and by using Corollary 2.1, we infer that $h(tX, Y) \in \Gamma(\mu)$ and our assertion follows.

We say that M is a *totally contact-umbilical submanifold* if there exists a normal vector field H such that the second fundamental form of M is given by

$$h(X, Y) = g(fX, fY)H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \quad \forall X, Y \in \Gamma(TM).$$

Then by using Corollary 2.1 we deduce

Corollary 2.6. *Each totally umbilical contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is a normal contact CR-submanifold.*

Next, suppose that $\{E_1, E_2, \dots, E_q\}$ is a local field of orthonormal frames, for the anti-invariant distribution $D^\perp$. Let us denote by A_i the shape operator with respect to $V_i = fE_i$, for $i = 1, 2, \dots, q$. Then, from Theorem 2.1 we have

Corollary 2.6. *The contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is normal if and only if*

$$(2.12) \quad A_i tX = tA_i X, \quad \forall X \in \Gamma(D).$$

By using (1.2), (1.3), (1.5) and (1.6), for $X \in \Gamma(TM)$, we deduce

$$(2.13) \quad \nabla_X E_i = tA_i X - B \nabla_X^\perp V_i + \beta g(X, E_i) \xi, \quad \forall X \in \Gamma(TM),$$

$$(2.14) \quad \nabla_X^\perp V_i = \omega(\nabla_X E_i) + Ch(X, E_i), \quad \forall X \in \Gamma(TM).$$

It is well known that X is a Killing vector field if and only if

$$g(\nabla_Z X, Y) + g(\nabla_Y X, Z) = 0, \quad \forall Y, Z \in \Gamma(TM).$$

If $Y, Z \in \Gamma(D)$, then we say that X is a D -Killing vector field.

Theorem 2.2. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is normal contact CR-submanifold if and only if E_i , $i = 1, 2, \dots, q$, are D -Killing vector fields.*

Proof. By using (2.13) we deduce

$$(2.15) \quad g(\nabla_X E_i, Y) + g(\nabla_Y E_i, X) = g(tA_i X - A_i tX, Y), \quad \forall X, Y \in \Gamma(D).$$

Now our assertion follows from (2.15) and Corollary (2.6).

The Lie derivative of t with respect to $Y \in \Gamma(TM)$ is given by

$$(2.16) \quad (\mathcal{L}_Y t)X = [Y, tX] - t[Y, X], \quad \forall X \in \Gamma(TM).$$

Now we define a new tensor field S^* by

$$(2.17) \quad S^*(Y, X) = (\mathcal{L}_Y t)X, \quad \forall X \in \Gamma(TM).$$

By using (2.3) we deduce

$$(2.18) \quad S(X, Y) = t^2[X, Y] - t[tX, Y] - Bh(tX, Y), \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp).$$

Next we deduce from (1.15) that $h(tX, Y) = Ch(X, Y) + \omega(\nabla_Y X)$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Thus we obtain $Bh(tX, Y) = -Q(\nabla_Y X)$ and by using (2.18) it follows that

$$(2.19) \quad S(X, Y) = tS^*(Y, X) + Q(\nabla_Y X).$$

Theorem 2.4. *Suppose that M is a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ such that*

$$(2.20) \quad Q(\nabla_X Y) = 0, \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp \oplus \{\xi\}).$$

Then M is normal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ if and only if

$$(2.21) \quad S^*(Y, X) = 0, \quad \forall X \in \Gamma(D \oplus \{\xi\}), Y \in \Gamma(D^\perp).$$

Proof. Suppose M be normal. By using Theorem 2.1 and relation (2.19) we obtain

$$(2.22) \quad Q(\nabla_Y X) = 0,$$

$$(2.23) \quad tS^*(Y, X) = 0,$$

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$. It follows from (2.20) and (2.22) that $Q([X, Y]) = 0$, what is equivalent to

$$(2.24) \quad Q([tX, Y]) = 0, \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp).$$

Now from (1.6), (2.16), (2.17), (2.23) and (2.24) we deduce (2.21). Conversely, suppose that (2.21) holds. Using (1.6), (2.16) and (2.20) we infer that

$$Q(\nabla_Y X) = 0.$$

Finally, our assertion follows from (2.19) and (2.25).

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Technical University "Gh. Asachi", Jassy,
Department of Mathematics

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CR-SUBVARIETĂȚI DE CONTACT NORMALE ALE UNEI VARIETĂȚI TRANS-SASAKI

(Rezumat)

Se introduce noțiunea de CR-subvarietate normală a unei varietăți trans-Sasaki și se stabilesc rezultate fundamentale privind proprietățile sale geometrice.

DIFFERENTIAL COMPARISON RESULTS OVER EXCEPTIONAL SETS

BY

M. TURINICI

Abstract. Some functional-differential comparison results involving discontinuous real valued functions are given, over (generally uncountable) exceptional sets.

Key words: zero Lebesgue measure set, Dini derivative/oscillation, maximality principle, comparison implication, mean value theorem.

1. Introduction. Let J be an open real interval, and u, v elements of $F(J, R)$ (the class of all functions from J to R). By a *differential comparison property* involving this couple on a sub-interval $[a, b]$ of J , we mean the possibility of deriving the implication

$$(1.1) \quad u(a) \leq v(a) \Rightarrow u(t) \leq v(t), \quad a \leq t \leq b,$$

from a set of global differential inequalities like

$$(1.2) \quad \Delta u(t) - T(u)(t) \leq \Delta v(t) - T(v)(t), \quad t \in [a, b] \setminus M.$$

Here, Δ is one of the *Dini derivative operators*, $T: F(J, R) \rightarrow F(J, R)$ a certain mapping and M , an (exceptional points) subset of $[a, b]$. For example, when u and v are continuous, M is denumerable and T is *contractive* — in a sense precised by Turinici [9] — the above property holds, according to Theorem 5 in the quoted paper; see also Kamont [7]. Note that, when $T(\cdot) \equiv 0$, this yields the Denjoy-Bourbaki mean value theorem (cf. Dieudonné [4, Ch. 8, § 5]). On the other hand, when u and v are absolutely continuous over J and M is of zero Lebesgue measure, the property in question is again holding (when $T(\cdot) \equiv 0$), as the result in Aumann [1, Ch. 7, § 1] shows; see also Gal [6].

Now, it is the main aim of this exposition to show that a further extension of this last statement — in the above described functional setting — is possible, even if the functions u and v are discontinuous over $[a, b]$. This will require the global differential inequalities (1.2) be completed with a set of (global) conditions involving

the *Dini oscillation operators* Ω ; details are given in Sec. 3. All preliminary material is collected in Sec. 2. Some particular aspects of the main results are being delineated in Sec. 4, and in Sec. 5 some applications of these facts to mean value theorems are given.

2. Preliminaries. Let $\mathbb{R} =]-\infty, \infty[$ stand for the real axis and m , the usual *Lebesgue measure* over it. Remember that the part A of $\mathbb{R}$ is of *zero Lebesgue measure* iff for each $\varepsilon > 0$, there is a denumerable family $\{]a_n, b_n[; n=1, 2, \dots\}$ of disjoint open intervals which covers A , such that its total measure be inferior to ε (that is, $\sum_n (b_n - a_n) < \varepsilon$). In particular, each *denumerable* part B of $\mathbb{R}$ has such a property.

Let in the following $J =]\alpha, \beta[$ (where $-\infty \leq \alpha < \beta \leq \infty$) be an open real interval. For each function $f: J \rightarrow \mathbb{R}$, denote

$$\Delta_+ f(t) = \liminf_{s \rightarrow t+} R_f(t, s), \quad \Delta^{(+)} f(t) = \limsup_{s \rightarrow t+} R_f(t, s),$$

where, by convention,

$$R_f(t, s) = \frac{f(t) - f(s)}{t - s}, \quad t, s \in J, \quad t \neq s.$$

These will be referred to as the *right inferior/superior Dini derivatives* of f at the considered point. Denote also

$$\Omega_{(+)} f(t) = \liminf_{s \rightarrow t+} (f(s) - f(t)), \quad \Omega^{(+)} f(t) = \limsup_{s \rightarrow t+} (f(s) - f(t)).$$

We name these, the *right inferior/superior Dini oscillations* of f at $t \in J$. The left counterpart of the introduced operators may now be constructed in an evident manner; so, we do not give details. Note that, when f admits a *right derivative* f'_+ over J , then

$$\Delta_{(+)} f(t) = \Delta^{(+)} f(t) = f'_+(t), \quad t \in J.$$

Likewise, when f is monotone over J , then

$$\Omega_{(+)} f(t) = \Omega^{(+)} f(t) = f(t+0) - f(t), \quad t \in J,$$

as it can be directly seen.

The following results involving these operators will be useful for us (see, for instance, Flett [5, Ch. 1, § 10] and respectively (from a more abstract context) Turinici [10]).

Proposition 1. *Let the part A of J be of zero Lebesgue measure. Then, a function $h = h_A$ from J into $\mathbb{R}$ may be constructed, with the properties*

$$(2.1) \quad h \text{ is increasing and continuous over } J,$$

$$(2.2) \quad h'_+(t) = \infty, \quad \text{for all } t \in A.$$

Proposition 2. *Let B be a denumerable part of J . Then, a function $k = k_B$ from J into $\mathbb{R}$ may be constructed so that*

$$(2.3) \quad k \text{ is increasing over } J,$$

$$(2.4) \quad k(t+0) - k(t) > 0, \quad \text{for all } t \in B.$$

A function $f: J \rightarrow \mathbb{R}$ will be said to be *upper semicontinuous from the left* over

J , when

$$(C_1) \quad \Omega^{(-)} f(t) \leq 0, \quad t \in J.$$

This condition may be also expressed as

$$(C_1') \quad f(t_n) \geq \lambda, \quad n \in \mathbb{N} \text{ and } t_n \uparrow t \text{ as } n \rightarrow \infty \text{ imply } f(t) \geq \lambda.$$

For example, (C_1) holds whenever

$$(C_2) \quad f \text{ is continuous over } J,$$

or, alternatively, whenever

$$(C_3) \quad f \text{ is increasing over } J.$$

As a variant of this, we may consider the weaker condition

$$(C_1^*) \quad \Omega_{(-)} f(t) \leq 0, \quad t \in J.$$

It will be referred to as f is *quasi upper semicontinuous from the left* over J . (We do not give characterizations of it, because (C_1^*) will be of no use for us.) Further, call the function f *lower semicontinuous from the right* over J , in case

$$(C_4) \quad \Omega_{(+)} f(t) \geq 0, \quad t \in J.$$

This may be also expressed as

$$(C_4') \quad f(t_n) \leq \lambda, \quad n \in \mathbb{N} \text{ and } t_n \downarrow t \text{ as } n \rightarrow \infty \text{ imply } f(t) \leq \lambda.$$

For example, (C_4) holds under (C_2) or (C_3) . As a variant of this, one may introduce the weaker condition

$$(C_4^*) \quad \Omega^{(+)} f(t) \geq 0, \quad t \in J.$$

It will be referred to as f is *quasi lower semicontinuous from the right* over J . (As before, we do not give characterizations of it, because (C_4^*) is of no use in the sequel.)

Concerning these notations, the following simple fact will be in effect for us.

Proposition 3. Let the function $f: J \rightarrow \mathbb{R}$ satisfy one of the conditions (C_1) (resp., (C_1')) and (C_4) (resp., (C_4')), and let the function $g: J \rightarrow \mathbb{R}$ be as in (C_2) or (C_3) . Then $f+g$ also satisfies the conditions accepted for f .

A function $f: J \rightarrow \mathbb{R}$ will be said to be *locally absolutely continuous* over J , when, for each $a, b \in J$ with $a < b$, the restriction of f to $[a, b]$ is absolutely continuous (in the usual sense). It follows directly from definition that, a condition like

$$(C_5) \quad f \text{ is locally absolutely continuous over } J$$

implies (C_2) ; the reciprocal is not valid, in general. We also note the useful property (cf. Flett [5, Ch. 1, § 10]):

Proposition 4. Let the function $f: J \rightarrow \mathbb{R}$ be as in (C_5) . Then, the image under f of any zero Lebesgue measure part of J is again of zero Lebesgue measure.

3. Main Results. Let $J =]\alpha, \beta[$ (where $-\infty \leq \alpha < \beta \leq \infty$) be an open real interval, and $f \mapsto T(f)$, a selfmap of $F(J, \mathbb{R})$. Letting $a, b \in J$ with $a < b$, we say the underlying operator T is *semi-contractive* over $[a, b]$, when a number $\rho > 0$ may be found with the property:

for each $\varepsilon > 0$, $r \in [a, b[$ and each couple $u, v \in F(J, \mathbb{R})$ with

$$(H_1) \quad v - u \text{ fulfils } (C_1)$$

the premises

$$(3.1) \quad u(r) < y(r) + \rho\varepsilon \quad \text{and} \quad u(t) > v(t), \quad r < t < s,$$

for some s in $]r, b[$ with $s - r < \rho$

imply

$$(3.2) \quad T(u)(r) < T(v)(r) + \varepsilon.$$

Suppose the considered operator T fulfils such a condition. Let the couple $u, v \in F(J, \mathbb{R})$ satisfy (H_1) as well as (for a certain denumerable part B of J)

$$(H_2) \quad \Delta u(t) \neq \infty, \quad \Delta v(t) \neq -\infty, \quad \text{for each } t \in J \setminus B,$$

$$(H_3) \quad \Omega u(t) \neq \infty, \quad \Omega v(t) \neq -\infty, \quad \text{for each } t \in B.$$

Here, Δ is any of the Dini derivative operators $\Delta^{(+)}$ or $\Delta_{(+)}$ and Ω , any of the Dini oscillation operators $\Omega^{(+)}$ or $\Omega_{(+)}$. Our first main result is

Theorem 1. *Suppose that a zero Lebesgue measure subset A of J including B may be found so as the specific conditions below be satisfied*

$$(H_4) \quad \Delta u(t) - T(u)(t) \leq \Delta v(t) - T(v)(t), \quad t \in [a, b] \setminus A,$$

$$(H_5) \quad \Omega u(t) \leq \Omega v(t), \quad t \in [a, b] \cap B.$$

Then, the implication (1.1) is necessarily holding.

Proof. Without any loss (passing to a $\rho/3$ -partition of $[a, b]$ if necessary) one may assume $b - a < \rho/3$. (Here, $\rho > 0$ is the number given by the semi-contractivity of the operator T over $[a, b]$.) Let $\varepsilon > 0$ be arbitrary fixed. We introduce a new function w , from J into $\mathbb{R}$, by the convention

$$w(t) = v(t) - u(t) + \varepsilon(t - a) + \delta(h(t) - h(a)) + \gamma(k(t) - k(a)), \quad t \in J.$$

Here, $h = h_A$ and $k = k_B$ are the functions from J to $\mathbb{R}$ given by Propositions 1 and 2, respectively. Also $\delta > 0$ and $\gamma > 0$ are constants determined by

$$\max\{\delta(h(b) - h(a)), \gamma(k(b) - k(a))\} < \frac{1}{3}\rho\varepsilon.$$

(This evidently requires (a fact we admit in the sequel) that

$$\min\{h(b) - h(a), k(b) - k(a)\} > 0,$$

i.e., the intersection between $[a, b[$ and A (respectively, B) is nonempty. For, otherwise, the hypotheses (H_4) and (H_5) remain without object.) According to the premise of (1.1), we admit $u(a) \leq v(a)$, and then, $w(a) \geq 0$. It is our intention to prove that $w(b) \geq 0$. To this end, let us introduce a new ordering $\leq$ over $[a, b]$ by the convention

$$t \leq s \quad \text{iff} \quad t \leq s \quad \text{and} \quad (w(t) \geq 0 \Rightarrow w(s) \geq 0).$$

By (H_1) plus Proposition 3, it is clear that w fulfils (C_1) . As a consequence of this, each ascending (modulo $\leq$) sequence in $[a, b]$ is a Cauchy sequence bounded from above (modulo $\leq$). So, by the ordering principle in Turinici [10], for the starting point a (in $[a, b]$) there must be a maximal (modulo $\leq$) point r in $[a, b]$ with $a \leq r$. Hence, in particular, $w(r) \geq 0$ (because $w(a) \geq 0$, by hypothesis); or, in other words,

$$(3.3) \quad v(r) + \varepsilon(r - a) + \delta(h(r) - h(a)) + \gamma(k(r) - k(a)) \geq u(r).$$

Suppose by contradiction $r \neq b$. For each t in $]r, b[$ we must have, by maximality,

$w(t) < 0$, $r < t < b$, or, equivalently,

$$(3.4) \quad v(r) + \varepsilon(t-a) + \delta(h(t) - h(a)) + \gamma(k(t) - k(a)) \geq u(t), \quad r < t < b.$$

Combining these relations yields

$$(3.5) \quad v(t) - v(r) + \varepsilon(t-r) + \delta(h(t) - h(r)) + \gamma(k(t) - k(r)) < u(t) - u(r),$$

for each t in $]r, b[$.

We now have several situations to discuss.

i) r is not an element of A . By (3.5), we have a relation like

$$(3.5') \quad v(t) - v(r) + \varepsilon(t-r) < u(t) - u(r), \quad r < t < b.$$

Hence, dividing by $t - r$ and taking $\limsup$ ($\liminf$) as $t \rightarrow r+$,

$$(3.6) \quad \Delta v(r) + \varepsilon \leq \Delta u(r).$$

This, in the perspective of (H_2) , shows both $\Delta u(r)$ and $\Delta v(r)$ are finite. On the other hand, by (3.3) plus the choice of δ and γ

$$u(r) < v(r) + \frac{1}{3}\rho\varepsilon + \delta(h(b) - h(a)) + \gamma(k(b) - k(a)) + \rho\varepsilon,$$

and, by (3.4), it is clear that

$$(3.7) \quad u(t) > v(t), \quad r < t < b.$$

As a consequence of these, one gets (by the semi-contractivity property of the operator T) a conclusion of the form (3.2), that is

$$-T(v)(r) < -T(u)(r) + \varepsilon.$$

Adding this to (3.6) (and remembering the finite character of $\Delta u(r)$, $\Delta v(r)$), a contradiction is obtained with respect to (H_4) .

ii) r belongs to $A \setminus B$. Again by (3.5),

$$v(t) - v(r) + \delta(h(t) - h(r)) < u(t) - u(r), \quad r < t < b.$$

So, dividing by $t - r$ and taking $\limsup$ ($\liminf$) as $t \rightarrow r+$, condition (H_2) will be contradicted.

iii) r belongs to B . As another consequence of (3.5),

$$v(t) - v(r) + \gamma(k(t) - k(r)) < u(t) - u(r), \quad r < t < b.$$

Hence, passing to $\limsup$ ($\liminf$) as $t \rightarrow r+$,

$$\Omega v(r) + \gamma(k(r+0) - k(r)) \leq \Omega u(r).$$

This, in the perspective of (H_3) , says both $\Omega u(r)$ and $\Omega v(r)$ are finite and $\Omega v(r) < \Omega u(r)$. Hence, (H_5) is contradicted.

From the above discussion it follows that, necessarily, $r=b$. Hence, $w(b) \geq 0$, or by (3.3),

$$u(b) \leq v(b) + \varepsilon(b-a) + \delta(h(b) - h(a)) + \gamma(k(b) - k(a)) < v(b) + \rho\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we must have $u(b) \leq v(b)$. This, plus the arbitrariness of b with respect to the condition $0 < b-a < \rho/3$, proves the implication (1.1) on such sub-intervals of $[a, b]$, and therefore — by the remark at the beginning — on the whole interval $[a, b]$. Hence the conclusion. Q.E.D.

Concerning this result, a basic aspect to be underlined in the intervention of the semi-open interval $[a, b[$ (hence, in particular, of the point a) in the specific assumptions $(H_4) + (H_5)$. So, it is natural to ask of whether or not is this removable.

The answer is positive and may be formulated under the lines below. Let again $f \vdash \vdash T(f)$ be a selfmap of $F(J, \mathbb{R})$ and $a, b \in J$ (with $a < b$) given points. We say the operator T is *weakly semi-contractive* over $[a, b]$ when a number $\rho > 0$ may be found with the property: for each $\varepsilon > 0$, r in $]a, b[$ and each couple $u, v \in F(J, \mathbb{R})$ with

$$(H_1') \quad v - u \text{ fulfils } (C_1) \text{ and } (C_4),$$

the premises below imply (3.2) again

$$(3.1') \quad v(r) \leq u(r) < v(r) + \rho\varepsilon \quad \text{and} \quad u(t) > v(t), \quad r < t < s,$$

for some s in $]r, b[$ with $s - r < \rho$.

Suppose the operator T fulfils this condition. Let the couple $u, v \in F(J, \mathbb{R})$ satisfy (H_1') as well as (H_2) (for a certain denumerable part B of J). Our second main result is

Theorem 2. *Suppose that a zero Labesgue measure part A of J including B may be found so as the specific condition below be satisfied*

$$(H_4') \quad \Delta u(t) - T(u)(t) \leq \Delta v(t) - T(v)(t), \quad t \in]a, b[\setminus A.$$

Then, the implication (1.1) is necessarily holding.

Proof. As before, one may assume without loss that $b - a < \rho/4$, where $\rho > 0$ is the number given by the weak semi-contractivity property of the operator T over $[a, b]$. Let $\varepsilon > 0$ be arbitrary fixed. We define a function $w: J \rightarrow \mathbb{R}$ by

$$w(t) = v(t) - u(t) + \varepsilon(t - a) + \delta(h(t) - h(a)) + \gamma(k(t) - k(a)) + \theta, \quad t \in J.$$

Here, $h = h_A$ and $k = k_B$ are the functions from J to $\mathbb{R}$ given by Proposition 1 and 2, respectively. Also, $\delta > 0$, $\gamma > 0$ and $\theta > 0$ are constants determined by

$$\max\{\delta(h(b) - h(a)), \gamma(k(b) - k(a)), \theta\} < \frac{1}{4}\rho\varepsilon.$$

According to the premise of (1.1) and the choice of θ , it is clear that $w(a) > 0$. We claim that the property below is true: $w(c) \geq 0$, for some c in $]a, b[$. For, if this would not be valid, then $w(t) - w(a) < -w(a) < 0$, $a < t < b$. So, passing to $\limsup$ as $t \rightarrow a+$, one gets a contradiction to (H_1') (the second part); hence the assertion. Now, assume without loss that $c \neq b$; since, otherwise, we finished the argument.

Let us introduce a new ordering $\leq$ on $[a, b]$ under the model of the first main result. By (H_1') (the first part) plus Proposition 3, the function w fulfils (C_1) . So, the already quoted author's maximality principle is again applicable to the structure $([a, b], \leq)$. As a consequence of this we have that, for the starting point c in $]a, b[$ (taken as before) there must be a maximal (modulo $\leq$ point r in $[a, b]$ with $c \leq r$. In particular, this shows that $w(r) \geq 0$ (since, by hypothesis, $w(c) \geq 0$); that is

$$(3.3') \quad v(r) + \varepsilon(r - a) + \delta(h(r) - h(a)) + \gamma(k(r) - k(a)) + \theta \geq u(r).$$

Suppose by contradiction $r \neq b$. We must have

$$(3.4) \quad v(t) + \varepsilon(t - a) + \delta(h(t) - h(a)) + \gamma(k(t) - k(a)) + \theta < u(t), \quad r < t < b.$$

Hence, combining these relations, (3.5) is again true.

It remains now to discuss the possibilities i) - iii) precised in the above argument. The alternative i) may be handled as before to produce a contradiction with respect to (H_4') . (Note that the left counterpart of the double inequality in (3.1) is a consequence of (3.7) plus (H_1') (the second part).) Likewise, the case ii) gives, in the way we already precised, a contradiction to (H_2) . Concerning the last possible

alternative iii), we have, by (3.5), that

$$v(t) - u(t) - (v(r) - u(r)) < -\gamma(k(t) - k(r)), \quad r < t < b.$$

Hence, passing to $\limsup$ as $t \rightarrow r+$, $\Omega^{(+)}(v-u)(r) \leq -\gamma(k(r+0) - k(r)) < 0$, a contradiction with respect to (H_1') (the second part). Summing up, $r=b$; or, in other words, $w(b) \geq 0$. This, in view of (3.3'), shows

$$u(b) \leq v(b) + \varepsilon(b-a) + \delta(h(b) - h(a)) + \gamma(k(b) - k(a)) + \theta < v(b) + \rho\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we must have $u(b) \leq v(b)$ and this, by the arbitrariness of b (with respect to the limiting condition $0 < b - a < \rho/4$) ends the argument. Q.E.D.

Concerning the general assumption (H_2) , used in both these statements, the natural question is to what extent can we replace the denumerable set B in this condition with a zero Lebesgue measure one. To give an appropriate answer we need the convention below. Let $f: T(f)$ be a selfmap of $F(J, \mathbb{R})$ and $[a, b]$ a subinterval of J . We say the operator T is *almost weakly semi-contractive* over $[a, b]$ when a number $\rho > 0$ may be found with the property: for each $\varepsilon > 0$, r in $]a, b[$ and each couple u, v in $F(J, \mathbb{R})$ with

$$(H_1'') \quad v - u \text{ fulfils } (C_5),$$

the premises in (3.1') imply (3.2). Suppose further that the underlying operator is endowed with such a property. We let a couple u, v in $F(J, \mathbb{R})$ which satisfies (H_1'') as well as (for a certain zero Lebesgue measure part A of J)

$$(H_2') \quad \Delta u(t) \neq \infty, \Delta v(t) \neq -\infty, \text{ for each } t \in J \setminus A.$$

As a completion of the statements above, our third main result is

Theorem 3. *Suppose that, for the zero Lebesgue measure subset A of J given by (H_2') , it is the case that (H_4') holds. Then the implication (1.1) is retainable.*

Proof. As already precised in the statements above, we may assume without any loss that $b - a < \rho/2$, where $\rho > 0$ is the constant introduced by the almost weak semi-contractivity of the operator T over $[a, b]$. Let $\varepsilon > 0$ be arbitrary fixed. We introduce a function w from J to $\mathbb{R}$ by the convention $w(t) = v(t) - u(t) + \varepsilon(t - a)$, $t \in J$. This function is, by (H_1'') , locally absolutely continuous over J . In addition (by the premise of (1.1)) we must have $w(a) \geq 0$. All we have to prove is that $w(b) \geq 0$. Suppose by contradiction $w(b) < 0$. Let η be a number subject to the restrictions $\max(-\rho\varepsilon/2, w(b)) < \eta < 0$, $\eta \notin w(A)$. (Note that, by Proposition 4, this is always possible.) Since, in particular, w is continuous over $[a, b]$, it follows by the Cauchy intersection theorem that an element r in $]a, b[$ may be found with the property $w(r) = \eta$ (hence, r does not belong to A), or, in other words,

$$(3.3'') \quad v(r) + \varepsilon(r - a) - \eta = u(r),$$

as well as $w(t) < \eta$, $r < t < b$, that is

$$(3.4') \quad v(t) + \varepsilon(t - a) - \eta < u(t), \quad r < t < b.$$

Hence, combining these facts, (3.5') is valid. In this case, the arguments developed in the first main result show that a contradiction is obtainable with respect to (H_4') . This proves that the assumption about $w(b)$ is unacceptable; or, in other words, $w(b) \geq 0$. But, the element b may be taken arbitrarily with respect to the condition $0 <$

$< b - a < \rho/2$. Hence, the conclusion follows on such subintervals of $[a, b]$, and therefore — by the same argument as in the first main result — on the whole interval $[a, b]$. The proof is thereby complete. Q.E.D.

4. Some Particular Aspects. The different variants of semi-contractivity conditions imposed upon the selfmap T of $F(J, \mathbb{R})$ are semi-local in nature, because they are expressed in terms of the arbitrarily small intervals $]r, s[$ of $[a, b]$. So, the most natural choice of these operators is the one expressed as

$$T(x)(t) = f(t, x(t)), \quad t \in J, \quad x \in F(J, \mathbb{R}),$$

where $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ is a function. It is our intention in what follows to study what happens with the results above under this choice of the underlying operator T ; for other aspects of the problem we refer to the paper by Turinici [9].

To begin with, we start by noting that the semi-contractivity property of this operator over $[a, b]$, may now be expressed as there exists a $\lambda > 0$ such that

$$(4.1) \quad t \in [a, b], \quad s_1 < s_2 + \varepsilon \Rightarrow f(t, s_1) < f(t, s_2) + \lambda \varepsilon.$$

This will be referred to as f being *semi-contractive* over $[a, b]$. Suppose f is endowed with such a property. Let u, v be a couple in $F(J, \mathbb{R})$ so as (H_1) be fulfilled, as well as $(H_2) + (H_3)$ (for a certain denumerable part B in J). As an immediate application of the first main result, we have

Corollary 1. Suppose that a zero Lebesgue measure subset A of J including B may be found so as

$$(4.2) \quad \Delta u(t) - f(t, u(t)) \leq \Delta v(t) - f(t, v(t)), \quad t \in [a, b] \setminus A,$$

$$(4.3) \quad \Omega u(t) \leq \Omega v(t), \quad t \in [a, b] \cap B.$$

Then, conclusion of Theorem 1 is retainable.

This Corollary extends a statement in this area due to the author [9]; for some abstract counterpart of these, we refer to the paper [10].

Further, the weak semi-contractivity property of the underlying operator over $[a, b]$ may be now expressed as: there is a $\lambda > 0$, such that

$$(4.1') \quad t \in]a, b[, \quad s_2 \leq s_1 < s_2 + \varepsilon \Rightarrow f(t, s_1) < f(t, s_2) + \lambda \varepsilon.$$

This will be referred to as f is weak *semi-contractive* over $[a, b]$. Suppose f is taken in accordance with such a property. Let u, v be a couple in $F(J, \mathbb{R})$ fulfilling (H_1') , as well as (H_2) (for a certain denumerable part B of J). As a direct application of the second main result, we have

Corollary 2. Suppose that a zero Lebesgue measure subset A of J including B may be found so that

$$(4.2') \quad \Delta u(t) - f(t, u(t)) \leq \Delta v(t) - f(t, v(t)), \quad t \in]a, b[\setminus A.$$

Then, conclusion of Theorem 2 is retainable.

This Corollary extends the basic statement in this area due to Bebernes and Meisters [2]. Precisely — unlike the quoted result — discontinuous functions are allowed in such a treatment.

An interesting particular case of these facts corresponds to the case of the function f being identically zero. Precisely, let the couple u, v of $F(J, \mathbb{R})$ be as in (H_1) ,

and assume that, for a certain denumerable part of J , $(H_2) + (H_3)$ are holding. We then have

C o r o l l a r y 3. *Suppose that a zero Lebesgue measure subset A of J including B may be found so as*

$$(4.2') \quad \Delta u(t) \leq \Delta v(t), \quad t \in [a, b] \setminus A,$$

$$(4.3') \quad \Omega u(t) \leq \Omega v(t), \quad t \in [a, b] \cap B.$$

Then, conclusion of Theorem 1 is retainable. Moreover, if at least one of the inequalities above is strict, then $u(b) < v(b)$.

P r o o f. The first part is clear, by Corollary 1. For the second one, denote for simplicity $w = v - u$. Let c, d in $[a, b]$ with $c < d$ be arbitrary fixed. It clearly follows by the implication (1.1) applied to the couple $(u + w(c), v)$, that $u(d) + w(c) \leq v(d)$ (that is, $w(c) \leq w(d)$). Hence, $t \mapsto w(t)$ is increasing over the interval $[a, b]$. Suppose by contradiction that $u(b) = v(b)$, or, in other words, $w(b) = 0$. In view of $0 \leq w(a) \leq w(b) = 0$, we necessarily have $w(t) = 0$ (that is, $u(t) = v(t)$), $a \leq t \leq b$. But, in this case, the relations $(4.2') + (4.3')$ become equalities. The contradiction at which we arrived shows $u(b) = v(b)$ is not acceptable. Hence the conclusion. Q.E.D.

Finally, let the couple u, v of $F(J, \mathbb{R})$ be as in (H_1) , and assume that, for a certain denumerable part B of J , condition (H_2) is holding. As an immediate consequence of Corollary 2, we have

C o r o l l a r y 4. *Suppose that there exists a zero Lebesgue measure subset A of J including B so that $(4.2')$ be valid, with $]a, b[$ in place of $[a, b]$. Then, conclusion of Corollary 3 is retainable.*

In particular, let the function $t \mapsto u(t)$ be constant over J . Then, Corollary 4 reduces to a statement in Gal [6], proved by a different technique; see also Aumann [1, Ch. 7, § I]. Corollary 3 extends some results in Turinici (op. cit.).

5. An Application. Let X be a real topological vector space. By a sublinear functional over X we mean any mapping $p: X \rightarrow \mathbb{R}$ with and properties

$$p(x+y) \leq p(x) + p(y), \quad p(\lambda x) = \lambda p(x), \quad \lambda \geq 0, \quad x, y \in X.$$

Suppose in the following that we fixed such a mapping p which, in addition, is continuous on all of X . Let $J =]\alpha, \beta[$ (where $-\infty \leq \alpha < \beta \leq \infty$) be an open real interval. We take a continuous function $G: J \rightarrow X$ and a function $g: J \rightarrow \mathbb{R}$ which fulfils (C_1) . In addition, assume there exists a denumerable part B of J such that

$$(K_1) \quad G_+^{\prime}(t) \text{ exists and } \Delta^{(+)} g(t) \neq -\infty, \text{ for all } t \text{ in } J \setminus B,$$

$$(K_2) \quad \Omega^{(+)} g(t) \neq -\infty, \text{ for all } t \text{ in } B.$$

As an immediate application of the statement above, the following vectorial mean value theorem (under inequality form) is available.

T h e o r e m 4. *Let the points a, b in J with $a < b$ be such that, for some zero*

Lebesgue measure part A of J including B , one has

$$(5.1) \quad p(G_+^f(t)) \leq \Delta^{(+)} g(t), \quad t \in [a, b] \setminus A,$$

$$(5.2) \quad 0 \leq \Omega^{(+)} g(t), \quad t \in [a, b] \cap B.$$

Then, necessarily, $p(G(b) - G(a)) \leq g(b) - g(a)$. Moreover, if at least one of the inequalities above is strict, then $p(G(b) - G(a)) < g(b) - g(a)$.

P r o o f. By the Hahn-Banach theorem (see, e.g., Cristescu [3, Ch. I, § 1]) there must be a continuous linear functional x^* over X with

$$(5.3) \quad x^*(x) \leq p(x), \quad \text{for all } x \in X,$$

$$(5.4) \quad x^*(G(b) - G(a)) = p(G(b) - G(a)).$$

Let the functions $u: J \rightarrow \mathbb{R}$, $v: J \rightarrow \mathbb{R}$, be introduced as

$$u(t) = x^*(G(t) - G(a)), \quad v(t) = g(t) - g(a), \quad t \in J.$$

By the admitted assumptions, $t \mapsto u(t)$ is continuous over J ; hence, by Proposition 3, the couple u, v fulfils (H_1) . In addition to this, $\Omega^{(+)} u(t) = 0$, for all $t \in J$. So, (H_3) holds (via (K_2)) and, by (5.2), it is clear that (4.3') is also valid. On the other hand, we have by (5.3),

$$\Delta^{(+)} u(t) = u_+^f(t) = x^*(G_+^f(t)) \leq p(G_+^f(t)), \quad t \in J \setminus B,$$

if we take (K_1) into account. This shows (H_2) is valid and, by (5.1), relation (4.2') will be also fulfilled. Summing up, Corollary 3 applies and, from this, conclusion is clear. Q.E.D.

As a counterpart of this, the following statement may be formulated. Let again $G: J \rightarrow X$ be a continuous function, and $g: J \rightarrow \mathbb{R}$, a function which satisfies $(C_1) + (C_4)$. Further, assume that for a certain denumerable part B of J , condition (K_1) is valid.

T h e o r e m 5. Let the points a, b in J with $a < b$ be such that, for some zero Lebesgue measure part A of J including B , it is the case that

$$(5.1') \quad p(G_+^f(t)) \leq \Delta^{(+)} g(t), \quad t \in]a, b[\setminus A.$$

Then, conclusions of Theorem 4 are retainable.

P r o o f. It is not hard to see that conditions of Corollary 4 are acceptable. Q.E.D.

In particular, when p is a norm on the ambient space X , the obtained facts may be viewed as an extension of the Denjoy-Bourbaki mean value theorem (see, e.g., Dieudonné [4, Ch. 8, § 5]) to the discontinuous case. On the other hand, when X is a (real) locally convex space and p is a continuous linear functional over X , then Theorem 5 gives in a standard way the main result in McLeod [8]. This will be discussed elsewhere.

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REZULTATE DE COMPARAȚIE DIFERENȚIALĂ
PESTE MULȚIMI EXCEPȚIONALE

(Rezumat)

Se dau o serie de rezultate de comparație de tip funcțional-diferențial relativ la funcții reale discontinue, peste mulțimi excepționale, în general nenumărabile.

MEMORANDUM

1. The purpose of this memorandum is to inform you of the results of the investigation conducted by the Special Agent in Charge, New York, on the subject of the activities of the German spy ring in New York City.

2. The investigation was conducted in accordance with the instructions of the Director, Federal Bureau of Investigation, dated 10-1-44.

3. The results of the investigation are as follows:

4. The German spy ring in New York City is composed of the following members:

5. The German spy ring in New York City is active in the following areas:

6. The German spy ring in New York City is active in the following areas:

7. The German spy ring in New York City is active in the following areas:

8. The German spy ring in New York City is active in the following areas:

9. The German spy ring in New York City is active in the following areas:

10. The German spy ring in New York City is active in the following areas:

REMARKS: The German spy ring in New York City is active in the following areas:

11. The German spy ring in New York City is active in the following areas:

POINTS FIXES COMMUNS POUR DES APPLICATIONS MULTIVOQUES QUI SATISFONT AUX INÉGALITÉS RATIONNELLES

PAR

NICOLETA NEGOESCU

Abstract. One proves some theorems on fixed points common for pairs and sets of multifunctions defined on complete metrical spaces satisfying the rational contractive inequalities (1) and (3).

Key words: fixed point theorem, multifunction, complete metrical space, contractive inequality.

1. Dans cette Note nous allons démontrer d'abord quelques théorèmes de point fixe commun pour des paires et pour des suites d'applications multivoques définies sur des espaces métriques complets en partant des théorèmes ressemblants obtenus par B. Fisher [1], V. Popa [5], et Nicoleta Negoescu [4], mais en employant les inégalités rationnelles contractives différentes (1) et (3).

Puis nous obtenons deux théorèmes de point fixe commun pour des paires et pour des suites d'applications multivoques définies sur un espace métrique avec deux métriques en partant des résultats obtenus par D. G. Maia [3], K. Iséki [2], I. A. Rus [7], [8], K. L. Singh [9] et V. Popa [5], [6], mais en employant aussi nos inégalités contractives (1) et (3).

2. Soit (X, d) un espace métrique. On note par $CB(X)$ l'ensemble de tous les sous-ensembles nonvides, fermés et bornés de X et par H la métrique Hausdorff induite par d sur $CB(X)$. Alors pour tous $A, B \in CB(X)$ on définit

$$D(x, A) = \inf_{y \in A} d(x, y),$$

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B); \sup_{y \in B} D(y, A) \right\},$$

$$\delta(A, B) = \sup \{ d(a, b) : a \in A, b \in B \}.$$

Observations. Si l'ensemble $A = \{a\}$, alors on écrit $\delta(A, B) = \delta(a, B)$. Si $\delta(A, B) = 0$, alors $A = B = \{a\}$ (v. le Lemme 1 [8], I. A. R u s).

Soient $A, B \in CB(X)$ et $k \in (1, \infty)$ donné. Dans la Note présente nous employerons l'affirmation suivante bien connue: pour tout $a \in A$ il existe $b \in B$ tel que $d(a, b) \leq kH(A, B)$.

Si $T: X \rightarrow X$ est une application multivoque on note par $F(T) = \{x \in X : x \in Tx\}$.

L e m m e. Soient (X, d) un espace métrique et les applications multivoques $S, T: X \rightarrow CB(X)$. Si l'inégalité

$$(1) \quad H^m(Sx, Ty) \leq c \frac{D^p(x, Sx) + D^p(y, Ty)}{\delta^{p-m}(x, Ty) + \delta^{p-m}(y, Sx)}$$

est vraie pour tous $x, y \in X$ tels que $\delta^{p-m}(x, Ty) + \delta^{p-m}(y, Sx) \neq 0$, où $0 < c < 1$, $m \geq 1$, $p \geq 2$, $m < p$ et $F(T) \neq \emptyset$, alors $F(S) \neq \emptyset$ et $F(S) = F(T)$.

D é m o n s t r a t i o n. Soit $u \in F(T)$, donc $u \in Tu$. Si $D(u, Su) \neq 0$, alors de l'inégalité (1) il suit

$$\begin{aligned} d^m(u, Su) &\leq H^m(Tu, Su) \leq c \frac{D^p(u, Tu) + D^p(u, Su)}{\delta^{p-m}(u, Tu) + \delta^{p-m}(u, Su)} \leq \\ &\leq c \frac{D^p(u, Su)}{\delta^{p-m}(u, Tu) + \delta^{p-m}(u, Su)} \leq c \frac{d^p(u, Su)}{d^{p-m}(u, Tu) + d^{p-m}(u, Su)} = c d^m(u, Su). \end{aligned}$$

Alors $d^m(u, Su) < c d^m(u, Su)$ ou $(1 - c)d^m(u, Su) < 0$ qui est une contradiction. Donc $D(u, Su) = 0$ et, Su étant un ensemble fermé, il suit $u \in Su$, ce qui signifie que nous avons montré que $F(T) \subset F(S)$.

De la même manière on montre que $F(S) \subset F(T)$ et donc $F(S) = F(T)$.

T h é o r è m e 1. Soient (X, d) un espace métrique complet et $S, T: X \rightarrow CB(X)$ deux applications multivoques qui satisfont à l'inégalité (1) pour tous les $x, y \in X$ tels que $\delta^{p-m}(x, Ty) + \delta^{p-m}(y, Sx) \neq 0$, où $0 < c < 1$, $m \geq 1$, $p \geq 2$ et $m < p$. Alors S et T ont des point fixes communs et $F(S) = F(T)$.

D é m o n s t r a t i o n. On choisit un nombre réel q tel que

$$(2) \quad 0 < q < (2c)^{-\frac{1}{m}}.$$

Soient les points $x_0 \in X$ et $x_1 \in Tx_0$. Alors il existe un point $x_2 \in Sx_1$ tel que $d(x_1, x_2) \leq qH(Tx_0, Sx_1)$. On suppose qu'on peut choisir les points $x_3, x_4, \dots, x_{2n-2}, x_{2n-1}, x_{2n}, \dots$ tels que

$$x_{2n-2} \in Sx_{2n-3}, \quad x_{2n-1} \in Tx_{2n-2}, \quad x_{2n} \in Sx_{2n-1},$$

$$d(x_{2n-2}, x_{2n-1}) \leq qH(Sx_{2n-3}, Tx_{2n-2}), \quad d(x_{2n-1}, x_{2n}) \leq qH(Tx_{2n-2}, Sx_{2n-1}).$$

D'abord on suppose que $\delta^{p-m}(x_{2n-1}, Tx_{2n-2}) + \delta^{p-m}(x_{2n-2}, Sx_{2n-1}) = 0$ pour $n = 1, 2, \dots$. Alors $x_{2n-1} = \{Tx_{2n-2}\} = x_{2n-2} = \{Sx_{2n-1}\}$ et $x_{2n-2} = x_{2n-1}$ est un point fixe commun pour S et T . De la même manière $\delta^{p-m}(x_{2n}, Sx_{2n-1}) + \delta^{p-m}(x_{2n-1}, Tx_{2n-2}) = 0$ pour $n = 1, 2, \dots$ implique $x_{2n-1} = x_{2n}$ est un point fixe commun pour S et T .

Maintenant on suppose que

$$\delta^{p-m}(x_{2n-1}, Tx_{2n-2}) + \delta^{p-m}(x_{2n-2}, Sx_{2n-1}) \neq 0,$$

$$\delta^{p-m}(x_{2n}, Sx_{2n-1}) + \delta^{p-m}(x_{2n-1}, Tx_{2n}) \neq 0, \text{ pour } n=1, 2, \dots$$

Alors de l'inégalité (1) il suit

$$\begin{aligned} d^m(x_{2n-1}, x_{2n}) &\leq q^m H^m(Tx_{2n-2}, Sx_{2n-1}) \leq \\ &\leq cq^m \frac{D^p(x_{2n-1}, Sx_{2n-1}) + D^p(x_{2n-2}, Tx_{2n-2})}{\delta^{p-m}(x_{2n-1}, Tx_{2n-2}) + \delta^{p-m}(x_{2n-2}, Sx_{2n-1})} \end{aligned}$$

et on obtient

$$d^m(x_{2n-1}, x_{2n}) \leq cq^m \frac{d^p(x_{2n-1}, x_{2n}) + d^p(x_{2n-2}, x_{2n-1})}{d^{p-m}(x_{2n-1}, x_{2n})}.$$

Donc $d^p(x_{2n-1}, x_{2n}) \leq cq^m(d^p(x_{2n-1}, x_{2n}) + d^p(x_{2n-2}, x_{2n-1}))$ et, en notant par $t = d(x_{2n-1}, x_{2n})/d(x_{2n-2}, x_{2n-1})$, on obtient $t^p \leq cq^m(t^p + 1)$.

On note par $f(t) = t^p(1 - cq^m) - cq^m$; $f'(t) = pt^{p-1}(1 - cq^m) > 0$ (car $1 - cq^m > 0$), et $f(0) = -cq^m < 0$, $f(1) = 1 - 2cq^m > 0$ (v. l'inégalité (2)).

Soit $k \in (0, 1)$ la racine de l'équation $f(t) = 0$. Alors $f(t) \leq 0$ pour $t \leq k$ et il suit

$$d(x_{2n-1}, x_{2n}) \leq kd(x_{2n-1}, x_{2n-2}) \text{ pour } n=1, 2, \dots$$

De la même manière on peut montrer que

$$d(x_{2n}, x_{2n+1}) \leq kd(x_{2n-1}, x_{2n}) \text{ pour } n=1, 2, \dots$$

En répétant le raisonnement antérieur on obtient

$$d(x_{n-1}, x_n) \leq k^n d(x_0, x_1) \quad \forall n \in \mathbb{N}^*.$$

Cette inégalité montre que la suite (x_n) est une suite de Cauchy et, car (X, d) est un espace métrique complet, il suit que (x_n) est une suite convergente et $\lim x_n = u \in X$.

Maintenant on suppose que $d(u, Tu) \neq 0$. Alors

$$\begin{aligned} D^m(x_{2n}, Tu) &\leq H^m(Sx_{2n-1}, Tu) \leq \\ &\leq c \frac{D^p(x_{2n-1}, Sx_{2n-1}) + D^p(u, Tu)}{\delta^{p-m}(x_{2n-1}, Tu) + \delta^{p-m}(u, Sx_{2n-1})} \leq c \frac{d^p(x_{2n-1}, x_{2n}) + D^p(u, Tu)}{d^{p-m}(x_{2n-1}, Tu) + d^{p-m}(u, x_{2n})}. \end{aligned}$$

Pour $n \rightarrow \infty$ dans l'inégalité antérieure il suit $D^m(u, Tu) \leq cD^m(u, Tu)$, une contradiction (car $c < 1$). Donc $D(u, Tu) = 0$. L'ensemble Tu étant fermé, il suit que $u \in Tu$ et du Lemme on obtient que $u \in Su$ et $F(T) = F(S)$.

Observation. Si dans le Théorème 1 on pose $T=S$, il suit un théorème analogue pour une seule application multivoque.

T h é o r è m e 2. Soient (X, d) un espace métrique complet et $S, T: X \rightarrow X$

deux applications telles que

$$d^m(Sx, Ty) \leq c \frac{d^p(x, Sx) + d^p(y, Ty)}{d^{p-m}(x, Ty) + d^{p-m}(y, Sx)}$$

pour tous $x, y \in X$ pour lesquels $d^{p-m}(x, Ty) + d^{p-m}(y, Sx) \neq 0$, où $0 < c < 1$, $m \geq 1$, $p \geq 2$, $m < p$.

Alors S et T ont des points fixes communs et $F(T) = F(S)$. Encore, si $d^{p-m}(y, Tx) + d^{p-m}(x, Sy) = 0$ implique $d(Tx, Sy) = 0$, alors les applications S et T ont un point fixe commun unique.

Démonstration. L'existence des points fixes communs suit du Théorème 1. Pour démontrer l'unicité on suppose que S et T ont aussi un deuxième point fixe u' , $u' \neq u$. Alors $d^{p-m}(u, Tu') + d^{p-m}(u', Su) = 0$ implique $d(Tu', Su) = 0$ et ainsi $u = Su$, $u' = Tu'$ et $Su = Tu'$ si et seulement si $u = u'$. Donc le point fixe commun de T et S est dans ce cas unique.

Observations. Il est évident qu'en dehors de la condition " $d^{p-m}(x, Tx) + d^{p-m}(y, Sy) = 0$ implique $d(Tx, Sy) = 0$ " le point fixe commun de S et T n'est pas nécessairement unique (v. Exemple p. 31 [1]).

Si $T = S$ alors du Théorème 2 on obtient un théorème analogue pour une seule autoapplication de l'espace métrique complet (X, d) .

Théorème 3. Soient (X, d) un espace métrique complet et $(T_n)_{n \in \mathbb{N}}$ une suite des applications multivoques de X dans $CB(X)$ telles que

$$(3) \quad H^m(T_1x, T_ny) \leq c \frac{D^p(x, T_1x) + D^p(y, T_ny)}{\delta^{p-m}(x, T_ny) + \delta^{p-m}(y, T_1x)}$$

est vraie pour tous $x, y \in X$, $n \geq 2$, $0 < c < 1$, $m \geq 1$, $p \geq 2$ et $m < p$ pour lesquels $\delta^{p-m}(x, T_ny) + \delta^{p-m}(y, T_1x) \neq 0$.

Alors la suite des applications multivoques $(T_n)_{n \in \mathbb{N}}$ a des points fixes communs et $F(T_1) = F(T_n)$.

La démonstration du Théorème 3 suit du Théorème 1 et du Lemme.

3. En partant des idées de D. G. Maia [3], K. Iséki [2], K. L. Singh [9], I. A. Rus [7], [8], V. Popa [5], [6], N. Negoescu [4] nous démontrerons deux théorèmes ressemblants aux Théorèmes 1 et 3 de cette Note pour des applications multivoques définies sur un espace métrique doué avec deux métriques.

Théorème 4. Soit X un espace métrique avec deux métriques d_1 et d_2 . Si les conditions suivantes sont satisfaites:

- i) $d_1(x, y) \leq d_2(x, y)$, $\forall x, y \in X$;
- ii) (X, d_1) est un espace métrique complet;
- iii) deux applications multivoques $T, S: X \rightarrow CB(X)$ sont ponctuellement fermées et ponctuellement bornées relativement aux métriques d_1 et d_2 ;
- iv) T ou S est semicontinue supérieurement par rapport à la métrique d_1 ;

les $x, y \in X$ tels que $\delta^{p-m}(x, Ty) + \delta^{p-m}(y, Sx) \neq 0$, où $0 < c < 1$, $m \geq 1$, $p \geq 2$ et $m < p$, alors S et T ont des points fixes communs et $F(T) = F(S)$.

Démonstration. De la même manière que dans le Théorème 1 on construit une suite $(x_n)_{n \in \mathbb{N}}$ en partant d'un point $\forall x_0 \in X$. La suite (x_n) est une suite de Cauchy par rapport à la métrique d_2 et $d_1(x, y) \leq d_2(x, y)$, $\forall x, y \in X$. Alors (x_n) est aussi une suite de Cauchy par rapport à la métrique d_1 et, car (X, d_1) est un espace métrique complet, il suit que (x_n) est convergente à $x \in X$. Mais T est semicontinue supérieurement et du Théorème 4 [6] il suit que T a le graphique fermé et alors $x_{2n+1} \in Tx_{2n}$ implique $x \in Tx$ et du Lemme on a $F(T) = F(S)$.

Théorème 5. Soit X un espace doué avec deux métriques d_1 et d_2 . Si les conditions suivantes sont satisfaites:

i) la suite (T_n) , $T_n: X \rightarrow CB(X)$ est formée par des applications multivoques ponctuellement fermées et ponctuellement bornées respectivement aux métriques d_1 et d_2 ;

ii) d_1, d_2, X et T_1 satisfont aux conditions i), ii) et iv) du Théorème 4;

iii) l'inégalité (3) est vraie pour tous $x, y \in X$ pour lesquels $\delta^{p-m}(x, T_n y) + \delta^{p-m}(y, T_n x) \neq 0$, où $n \geq 2$, $0 < c < 1$, $m \geq 1$, $p \geq 2$, $m < p$, alors la suite des applications multivoques (T_n) a des points fixes communs et $F(T_1) = F(T_n)$.

La démonstration de ce théorème suit du Théorème 4 et du Lemme.

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Université Technique "Gh. Asachi", Jassy,
Chaire de Mathématiques

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PUNCTE FIXE COMUNE PENTRU MULTIFUNCȚII CARE SATISFAC INEGALITĂȚI RAȚIONALE

(Rezumat)

Se demonstrează câteva teoreme de punct fix comun pentru perechi și pentru șiruri de multifuncții definite pe spații metrice complete care satisfac inegalitățile contractive raționale (1) sau (3):

ON THE SYSTEM $\dot{x}=A(t)x$ WITH $A(t)$ SYMMETRIC OR ANTISYMMETRIC

BY

ADRIAN CORDUNEANU and GH. MOROȘANU

Abstract. One studies how the symmetry or the antisymmetry of the n order real continuous matrix-function $A(t)$ of a classical linear differential system (1) is reflected in the properties of the fundamental matrix of this system. The main result is given in Theorem 1.

Key words: linear differential system, matrix-function, symmetric and antisymmetric matrix, fundamental matrix.

0. In this Note, we consider the classical linear differential system

$$(1) \quad \dot{x}(t) = A(t)x(t), \quad t \geq 0,$$

where $A=A(t)$ is a real and continuous matrix-function of order n and $x=x(t)$ is a vector-function of size $n \times 1$ (a column vector). Our aim is to study how the symmetry or the antisymmetry of $A(t)$ is reflected in the properties of the fundamental matrix of system (1).

1. First, we remark that in general, there is not a direct connection between the symmetry of $A(t)$ and the existence of some symmetric fundamental matrices of system (1). The next example shows that there exist nonsymmetric matrices $A(t)$, such that (1) admits symmetric fundamental matrices even in the simplest case when $A = \text{const}$.

Example 1. The system (with $n=2$)

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \text{nonsymmetric}$$

admits the fundamental matrix

$$X(t) = \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} = \text{symmetric}.$$

On the other hand, there exist symmetric matrices $A(t)$, such that system (1) has

no symmetric fundamental matrix, as the following simple example shows.

Example 2. Let us consider the system (with $n=2$)

$$\begin{cases} \dot{x} = tx + y \\ \dot{y} = x \end{cases}, \quad A(t) = \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} = \text{symmetric},$$

from which it follows $\dot{x} - tx - 2x = 0$. Seeking for this equation solution under the form of a power series in the variable t , we obtain for the system two linear independent solutions, namely

$$(i) \quad \begin{cases} x_1(t) = 1 + t^2 + \frac{1}{3}t^4 + \dots + \frac{1}{(2n-1)!!}t^{2n} + \dots \\ y_1(t) = t + \frac{1}{3}t^3 + \dots + \frac{1}{(2n+1)!!}t^{2n+1} + \dots \end{cases}$$

and respectively

$$(ii) \quad \begin{cases} x_2(t) = t + \frac{1}{2}t^3 + \dots + \frac{1}{(2n)!!}t^{2n+1} + \dots \\ y_2(t) = 1 + \frac{1}{2}t^2 + \dots + \frac{1}{(2n+2)!!}t^{2n+2} + \dots \end{cases}$$

Imposing to the matrix

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha, \beta, \gamma, \delta = \text{const.}$$

the condition of symmetry, it easily follows $\alpha, \beta, \gamma, \delta = 0$ which, of course, is impossible for a fundamental matrix. This example shows that to the symmetry of $A(t)$ it must be added a supplementary condition to ensure the existence of a symmetric fundamental matrix for the corresponding differential system.

2. The main result of this paper is the following theorem, which seems to be new.

Theorem 1. Assume that, in (1), $A = A(t)$ is continuous for $t \geq 0$. Then, the following two conditions are equivalent:

- (a) $A(t)$ is symmetric and $A(t)A(s) = A(s)A(t)$ for every $t, s \geq 0$;
- (b) there exists at least one symmetric fundamental matrix $X(t)$, such that $X(t)X(s) = X(s)X(t)$ for every $t, s \geq 0$.

Proof. Suppose that (a) is true. Let us consider the matrix

$$(2) \quad X(t) = \exp \left[\int_0^t A(s) ds \right], \quad t \geq 0.$$

Denoting

$$(3) \quad B(t) = \int_0^t A(s) ds, \quad t \geq 0,$$

we have

$$(4) \quad X(t) = I + \frac{1}{1!} B(t) + \frac{1}{2!} B^2(t) + \dots + \frac{1}{k!} B^k(t) + \dots,$$

where I stands for the identity matrix of order n .

Using the equalities $\dot{B}(t) = A(t)$ and $A(t)A(s) = A(s)A(t)$ we obtain $B(t)\dot{B}(t) = \dot{B}(t)B(t)$ for $t \geq 0$. This implies

$$(5) \quad \frac{d}{dt} B^k(t) = k \dot{B}(t) B^{k-1}(t), \quad t \geq 0, \quad k = \text{integer} \geq 1$$

and then, from (4), it follows

$$(6) \quad \dot{X}(t) = \dot{B}(t) \left[I + \frac{1}{1!} B(t) + \frac{1}{2!} B^2(t) + \dots \right] = A(t)X(t)$$

for $t \geq 0$, i.e. $X(t)$ is a fundamental matrix for (1). The symmetry of $X(t)$ follows from the expansion (4) and the symmetry of $B(t)$. Moreover, from the equality

$$(7) \quad B(t)B(s) = \int_0^t \int_0^s A(t_1)A(s_1) dt_1 ds_1 = \int_0^s \int_0^t A(s_1)A(t_1) ds_1 dt_1 = B(s)B(t),$$

holding for every $t, s \geq 0$, it also follows $X(t)X(s) = X(s)X(t)$, because both terms are equal to $\exp(B(t) + B(s))$. Consequently, the implication (a) $\Rightarrow$ (b) is proved.

Conversely, suppose that (b) is true. Starting from

$$(8) \quad A(t) = \dot{X}(t)X^{-1}(t), \quad t \geq 0$$

we shall prove that $A(t)$ is symmetric. Indeed, from $\dot{X}(t)X(s) = X(s)\dot{X}(t)$ for $t, s \geq 0$ and $t \neq s$, it follows

$$(9) \quad \dot{X}(t)X^{-1}(s) = X^{-1}(s)\dot{X}(t), \quad t, s \geq 0, \quad t \neq s.$$

Making $s \rightarrow t$ fixed, in the preceding equation, we also have

$$(10) \quad \dot{X}(t)X^{-1}(t) = X^{-1}(t)\dot{X}(t), \quad t \geq 0.$$

From (8) and (10) and taking into account that $X^{-1}(t), \dot{X}(t)$ are symmetric, it easily follows

$$(11) \quad A^T(t) = (X^{-1}(t))^T (\dot{X}(t))^T = X^{-1}(t) \dot{X}(t) = \dot{X}(t) X^{-1}(t) = A(t)$$

for $t \geq 0$; thus $A(t)$ is symmetric. By A^T we have denoted the transpose of matrix A . To prove that $A(t)A(s) = A(s)A(t)$ for $t, s \geq 0$, we shall make use of (8), (9), (10) and of the obvious equation $\dot{X}(t)\dot{X}(s) = \dot{X}(s)\dot{X}(t)$, $t, s \geq 0$. From

$$(12) \quad A(t)A(s) = \dot{X}(t)X^{-1}(t)\dot{X}(s)X^{-1}(s), \quad t, s \geq 0$$

and taking into account that every of the four matrices of the right hand side of the preceding equation commutes with all the other, we obtain

$$(13) \quad A(t)A(s) = \dot{X}(s)X^{-1}(s)\dot{X}(t)X^{-1}(t) = A(s)A(t), \quad t, s \geq 0.$$

This shows that the implication (b) $\Rightarrow$ (a) is true, what closes the proof of Theorem 1.

3. Now, we can ask about the class of continuous matrix-functions $A(t)$, defined for $t \geq 0$, which satisfy the condition of commutativity $A(t)A(s) = A(s)A(t)$ for

$t, s \geq 0$. A large set of such matrices is given by the expression

$$(14) \quad A(t) = a_1(t)A_1 + a_2(t)A_2 + \dots + a_m(t)A_m, \quad t \geq 0,$$

where $a_i(t), i = \overline{1, m}$ are continuous scalar functions and $A_i, i = \overline{1, m}$ are constant matrices of order n , satisfying $A_i A_j = A_j A_i$ for every $i, j = \overline{1, m}$. In particular, every matrix of the form

$$(15) \quad A(t) = a(t)I + b(t)B, \quad t \geq 0,$$

where $a(t), b(t)$ are continuous scalar functions and B is a constant matrix of order n , satisfies the condition under discussion. Moreover, if in (14) $A_i, i = \overline{1, m}$ are symmetric, then $A(t)$ is also symmetric.

Another example of matrix $A(t)$ satisfying the condition of commutativity is given by the formula

$$(16) \quad A(t) = (tI - B)^{-1}, \quad t \geq 0,$$

where B is a constant matrix of order n , having no eigenvalue on the real semiaxis $t \geq 0$. $A(t)$ is the so-called resolvent matrix of B . Multiplying matrix $A(t)$ given by (16) by the scalar function $P_B(t) = \det(tI - B)$, we obtain the matrix $\tilde{A}(t) = \text{adj}(tI - B)$, which satisfies $\tilde{A}(t)\tilde{A}(s) = \tilde{A}(s)\tilde{A}(t)$ for every complex numbers t and s .

Regarding the class of matrix functions, under discussion here, it may be proved a simple result, under the form of the following

Proposition 1. *If $A(t)$ satisfies the condition $A(t)A(s) = A(s)A(t)$ for $t, s \geq 0$ and $f(\lambda)$ is an analytical function over the whole complex field, then*

$$(17) \quad B(t) = f(A(t)), \quad t \geq 0$$

satisfies the condition $B(t)B(s) = B(s)B(t)$ for every $t, s \geq 0$. Moreover, if $A(t)$ is symmetric, then $B(t)$ is also symmetric.

Remark 1. From the proof of Theorem 1, it also follows that dropping out the condition of symmetry for the matrix $A(t)$, we obtain the equivalence of the propositions

(a') $A(t)A(s) = A(s)A(t)$ for every $t, s \geq 0$;

(b') *there exists at least one fundamental matrix $X(t)$ of (1), such that $X(t)X(s) = X(s)X(t)$ for every $t, s \geq 0$.*

4. In the case when the continuous matrix-function $A(t)$, defined for $t \geq 0$, is antisymmetric ($A^T(t) \equiv -A(t)$), we can prove a simple characterization of system (1), under the form of a more or less classical result (see, for example, [2]).

Proposition 2. *The conditions*

(i) $A(t)$ is antisymmetric for $t \geq 0$

and

(ii) *there exists orthogonal fundamental matrices of the system (1) are equivalent.*

Proof. Assume $A^T(t) = -A(t)$ for $t \geq 0$ and let $X(t)$ be the fundamental matrix of (1), satisfying the initial condition $X(0) = I$. The solution of (1), satisfying the initial condition $x(0) = c$, is given by $x(t) = X(t)c$. Here c is a real constant vector of size $n \times 1$. Since we have

$$(18) \quad x^T(t)\dot{x}(t) = x^T(t)A(t)x(t) = 0, \quad t \geq 0$$

and because $(x^T(t)x(t))' = 2x^T(t)\dot{x}(t) = 0$ for $t \geq 0$, it follows

$$(19) \quad x^T(t)x(t) = \text{const.} = c^T c, \quad t \geq 0.$$

This implies

$$(20) \quad c^T X^T(t)X(t)c = c^T c, \quad t \geq 0.$$

Taking into account that c is an arbitrary vector, it follows that $X^T(t)X(t) = I$ for $t \geq 0$, i.e. $X(t)$ is an orthogonal matrix for every $t \geq 0$.

Conversely, assume that $X(t)$ is an orthogonal fundamental matrix of (1). For the solution $x(t) = X(t)c$, we have again that (20), (19) and (18) hold. If in the second term of the equation (18), we put $x(t) = X(t)c$, it follows

$$(21) \quad c^T X^T(t)A(t)X(t)c = 0, \quad t \geq 0.$$

For fixed $t \geq 0$ and denoting $y = X(t)c$, we obtain

$$(22) \quad y^T A(t)y = 0.$$

Because here y is an arbitrary column vector, it follows that $A(t)$ is antisymmetric for every $t \geq 0$. This ends the proof.

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Technical University "Gh. Asachi", Jassy,
Department of Mathematics
and
University "Al. I. Cuza", Jassy,
Faculty of Mathematics

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ASUPRA SISTEMULUI $\dot{x} = A(t)x$ CU $A(t)$ SIMETRICĂ SAU ANTISIMETRICĂ

(Rezumat)

Se studiază în ce măsură simetria sau antisimetria funcției matriceale reale continue $A(t)$ de ordinul n , prezintă în sistemul diferențial liniar clasic (1), este reflectată în proprietățile matricei fundamentale a acestui sistem. Rezultatul central al studiului este prezentat în Teorema 1.

JORDAN BASES AND LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

BY

I. ONCIULESCU

Abstract. One considers the n -order linear differential equation (1) and the system of linear differential equations (2), both with constant coefficients. One obtains the Jordan bases in the space $C_n(\cdot)$ corresponding to the system (2) matrix and also a basis in the vectorial space of the solutions to system (2), defined on $\mathbb{R}$ and with values in C_n .

Key words: linear differential equation, system of linear differential equations, linear space of solutions, Jordan basis, eigenvectors.

1. Introduction. We consider the differential linear equation of n -order ($n > 1$)

$$(1) \quad x^n(t) + s_1 x^{(n-1)}(t) + \dots + a_n x(t) = 0, \quad t \in \mathbb{R},$$

and the linear differential system

$$(2) \quad \mathbf{x}'(t) = A \mathbf{x}(t),$$

where $a_i \in \mathbb{R}$ ($i = \overline{1, n}$) are constants,

$$(3) \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

are the system matrix and the unknown vector-function.

We denote $S(R, \mathbb{C})$ the n -dimensional linear space of solutions x defined on $\mathbb{R}$ with values in $\mathbb{C}$ (the complex numbers field) of the equation (1) and $T(R, \mathbb{C})$ the n -dimensional linear space of the vector-solutions $\mathbf{x}$ defined on $\mathbb{R}$ with values in the space C_n of the system

(*) $(C_k, k \geq 1, \text{ represents the set } C_k = \{v = (v_1 v_2 \dots v_k)^T, v_i \in \mathbb{C}, i = \overline{1, k}\})$

as usual linear complex space).

It is known that the spaces $S(\mathbb{R}, \mathbb{C})$ and $T(\mathbb{R}, \mathbb{C}_n)$ are isomorphic.

In this paper we shall first find a Jordan basis in $\mathbb{C}_n$, corresponding to the matrix A . Then, using such a basis, we shall give a basis in $T(\mathbb{R}, \mathbb{C}_n)$ [8], [6], and by intermediate of an isomorphism between $S(\mathbb{R}, \mathbb{C})$ and $T(\mathbb{R}, \mathbb{C}_n)$ we shall determine a basis in $S(\mathbb{R}, \mathbb{C})$.

2. A Jordan Basis in $\mathbb{C}_n$ Corresponding to the Matrix (3). Let λ_0 be an eigenvalue of the matrix A and let

$$S_{\lambda_0}(A) = \{v, v \in \mathbb{C}_n, (A - \lambda_0 I_n)v = 0\}$$

(I_n is the n -order unit matrix) be the corresponding proper subspace.

We observe that the determinant of the $n - 1$ -order quadratic matrix

$$(4) \quad B(\lambda_0) = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ -\lambda_0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\lambda_0 & 1 \end{vmatrix}$$

is a $n - 1$ order minor of the matrix $A - \lambda_0 I_n$ and $\det B(\lambda_0) = 1$. From these, it results that [4], [5] $\dim S_{\lambda_0}(A) = 1$.

As a consequence of this fact, in a Jordan basis in $\mathbb{C}_n$ (of the matrix A) it exists one series of eigenvectors and associated vectors, and, if n_0 is the multiplicity of λ_0 as root of the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$, such a series has n_0 vectors [5], [7]. In the following, we shall find such a series, therefore, a vector-system $\{y_1, \dots, y_{n_0}\} \subset \mathbb{C}_n$ which satisfies the compatible system

$$(5) \quad (A - \lambda_0 I_n)y_1 = 0, \quad (A - \lambda_0 I_n)y_2 = y_1, \dots, (A - \lambda_0 I_n)y_{n_0} = y_{n_0-1}, \quad y_1 \neq 0.$$

Consider the vector $e = (1 \ 0 \ \dots \ 0)^T \in \mathbb{C}_{n-1}$, and, if $v = (v_1 \ v_2 \ \dots \ v_n)^T \in \mathbb{C}_n$ we introduce the following vectors belonging to $\mathbb{C}_{n-1}$:

$$v[n] = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}, \quad v[1] = \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}.$$

If $\{y_1, \dots, y_{n_0}\}$ of (5) are of the form $y_k = (y_{k1} \ y_{k2} \ \dots \ y_{kn})^T$, $k = \overline{1, n_0}$, then, the system

$$(6_1) \quad B(\lambda_0)y_1[1] = \lambda_0 y_{11} e$$

is a subsystem of the system

$$(5_1) \quad (A - \lambda_0 I_n)y_1 = 0$$

and, for a p , $1 < p \leq n_0$, the system

$$(6_p) \quad B(\lambda_0)y_1[1] = \lambda_0 y_{11} e, \quad B(\lambda_0)y_k[1] = \lambda_0 y_{k1} e + y_{k-1}[n], \quad k = \overline{2, p},$$

is a subsystem of the system

$$(5_p) \quad (A - \lambda_0 I_n) y_1 = 0, \quad (A - \lambda_0 I_n) y_k = y_{k-1}, \quad k = \overline{2, p}.$$

It is immediate the fact that the algebraic system (6_p) ($1 \leq p \leq n_0$) is compatible.

The set of solutions with $y_1 \neq 0$ of the system (6_p) is

$$(7) \quad S_p = \{(y_1, y_2, \dots, y_p), \quad y_1 = \alpha_1 h_1, \quad y_2 = \alpha_2 h_1 + \alpha_1 h_2, \dots$$

$$\dots, y_p = \alpha_p h_1 + \alpha_{p-1} h_2 + \dots + \alpha_2 h_{p-1} + \alpha_1 h_p, \quad \alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{C}, \quad \alpha_1 \neq 0\},$$

where

$$(8) \quad h_1 = \begin{bmatrix} 1 \\ \lambda_0 B^{-1}(\lambda_0) e \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 \\ B^{-1}(\lambda_0) h_1[n] \end{bmatrix}, \quad \dots, \quad h_p = \begin{bmatrix} 0 \\ B^{-1}(\lambda_0) h_{p-1}[n] \end{bmatrix}$$

(here we have denoted a vector $v = (v_1 \ v_2 \ \dots \ v_n)^T \in \mathbb{C}_n$, in the form $v = (v_1 \ v[1])^T$). We shall prove, by induction, that the set S_p , of (7), represents, also, the set of all solutions, with $y_1 \neq 0$, of the system (5_p) , $1 \leq p \leq n_0$. The vector $y_1 = \alpha_1 h_1$, $\alpha_1 \neq 0$ verifies the system (6_1) . Because $\det(A - \lambda_0 I_n) = 0$ and $\det B(\lambda_0) = 1$, (6_1) is a principal subsystem of the system (5_1) . Consequently, the set S_1 is the set of all solutions nonzero of the system (5_1) .

Remark 1. By removing the first column of the matrix $A - \lambda_0 I_n$, we obtain a matrix U . If v is a vector of $\mathbb{C}_n$, then, the matrix obtained bordering the matrix U with the vector v (as last column) will be denoted by $[U, v]$.

Now, we prove that

$$(9) \quad \det[U, h_1] = 0,$$

and that S_2 is the set of all solutions of the system (5_2) . For this purpose we consider the system

$$(10) \quad (A - \lambda_0 I_n) y_2 = \alpha_1 h_1.$$

This system is a compatible system for a value $\alpha_1 \neq 0$, because the system (5_2) is a compatible system and S_1 is the set of all solutions of the system (5_1) . Using the well known Kronecker -Capelli theorem, it results, first, that $\det[U, \alpha_1 h_1] = 0$, and then, taking into account that $\alpha_1 \neq 0$, the equality (9). From here, it results that the system

$$(11) \quad (A - \lambda_0 I_n) y_2 = h_1$$

is a compatible one. A principal one of this system is $B(\lambda_0) y_2[1] = h_1[n]$ and, of (8), h_2 is a solution of this system. It results so, that h_2 is a solution of the system (11), and consequently, (h_1, h_2) verifies the system (5_2) . Using this fact, one can easily verify that every (y_1, y_2) belonging to S_2 is a solution of the system (5_2) .

Obviously, every solution (y_1, y_2) of the system (5_2) is a solution of the system (6_2) . So, it is proved that S_2 is the set of all solutions with $y_1 \neq 0$ of the system (5_2) . Further more, we suppose that

$$(H_1) \quad \det[U, h_1] = \det[U, h_2] = \dots = \det[U, h_{p-1}] = 0, \quad 1 \leq p \leq n_0 - 1,$$

and

$$(H_2) \quad S_p \text{ is the set of all solutions with } y_1 \neq 0 \text{ of the system } (5_p).$$

Under these hypotheses we shall prove, first, that

$$(12) \quad \det[U, \mathbf{h}_p] = 0$$

and, after that, S_{p+1} is the set of all solutions of the system (5_{p+1}) .

We consider the system

$$(13) \quad (A - \lambda_0 I_n) \mathbf{y}_{p+1} = \alpha_p \mathbf{h}_1 + \alpha_{p-1} \mathbf{h}_2 + \dots + \alpha_1 \mathbf{h}_p.$$

The system (13) is a compatible one for some values $\alpha_1, \dots, \alpha_p$, $\alpha_1 \neq 0$, because the system (5_{p+1}) is a compatible one and because we have supposed (H_2) . It results, as well as in the previous reasoning, that

$$(14) \quad \det[U, \alpha_p \mathbf{h}_1 + \alpha_{p-1} \mathbf{h}_2 + \dots + \alpha_1 \mathbf{h}_p] = 0.$$

From (14), taking into account (H_2) and the condition $\alpha_1 \neq 0$, it results the equality (12). Of (12) it results that the system

$$(15) \quad (A - \lambda_0 I_n) \mathbf{y}_{p+1} = \mathbf{h}_p$$

is a compatible one. As well as in the similar problem concerning the vector $\mathbf{h}_2$, it results that $\mathbf{h}_{p+1}$ verifies (15), hence $(\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_p, \mathbf{h}_{p+1})$ is a solution of the system (6_{p+1}) . If we consider now, $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{p+1}\} \in S_{p+1}$, we can write

$$\begin{aligned} (A - \lambda_0 I_n) \mathbf{y}_{p+1} &= \alpha_{p+1} (A - \lambda_0 I_n) \mathbf{h}_1 + \alpha_p (A - \lambda_0 I_n) \mathbf{h}_2 + \dots + \alpha_1 (A - \lambda_0 I_n) \mathbf{h}_{p+1} = \\ &= \alpha_p \mathbf{h}_1 + \alpha_{p-1} \mathbf{h}_2 + \dots + \alpha_1 \mathbf{h}_p = \mathbf{y}_p. \end{aligned}$$

It is proved so that the set S_{p+1} is one of solutions of the system (5_{p+1}) . Because S_{p+1} is the set of all solutions of the system (6_{p+1}) which is a subsystem of the system (5_{p+1}) , it results that S_{p+1} is the set of all solutions of the system (5_{p+1}) .

Thus, it is proved the following

L e m m a. Let be the matrix (3) and let λ_0 be a root of the characteristic polynomial $p(\lambda) = \det(\lambda J_n - A)$ with the multiplicity n_0 . Then the set S_{n_0} of (7) represents all series of eigenvectors and associated vectors, in $\mathbb{C}_n$, with n_0 vectors, corresponding to the eigenvalue λ_0 , of the matrix A .

Remark 2. An analogous result as in Lemma is true and for a certain matrix $A = (a_{ij})_{n \times n}$, $a_{ij} \in \mathbb{C}$, if λ_0 is a root of the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$ and $\text{rank}(\lambda_0 I_n - A) = n - 1$.

An immediate consequence of the Lemma and of a Theorem of [5], [7] is the following

T h e o r e m 1. Let be the matrix (3) and let $\lambda_1, \lambda_2, \dots, \lambda_p$, $\lambda_i \neq \lambda_j$, $i \neq j$, $i = \overline{1, p}$ be the roots of the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$, with the multiplicities $n_1, n_2, \dots, n_p$, respectively.

Then the vector system

$$(16) \quad \{\mathbf{y}_1^1, \dots, \mathbf{y}_{n_1}^1, \mathbf{y}_1^2, \dots, \mathbf{y}_{n_2}^2, \dots, \mathbf{y}_1^p, \dots, \mathbf{y}_{n_p}^p\},$$

where

$$(17) \quad \mathbf{y}_j^i = \alpha_j^i \mathbf{h}_1^i + \alpha_{j-1}^i \mathbf{h}_2^i + \dots + \alpha_1^i \mathbf{h}_j^i, \quad \alpha_j^i \in \mathbb{C}, \quad i = \overline{1, p}, \quad j = \overline{1, n_i}, \quad \alpha_1^i \neq 0,$$

$$(18) \quad \mathbf{h}_1^i = \begin{bmatrix} 1 \\ \lambda_i B^{-1}(\lambda_i) \mathbf{e} \end{bmatrix}, \quad \mathbf{h}_2^i = \begin{bmatrix} 0 \\ B^{-1}(\lambda_i) \mathbf{h}_1^i[n] \end{bmatrix}, \quad \dots, \quad \mathbf{h}_{n_i}^i = \begin{bmatrix} 0 \\ B^{-1}(\lambda_i) \mathbf{h}_{n_i-1}^i[n] \end{bmatrix},$$

$$B(\lambda_i) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -\lambda_i & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\lambda_i & 1 \end{pmatrix}, \quad i = \overline{1, p}, \quad e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}_{n-1},$$

is a Jordan basis in $\mathbb{C}_n$, corresponding to the matrix A .

Remark 3. A particular Jordan basis in $\mathbb{C}_n$, corresponding to the matrix A is the vector system

$$(19) \quad \{h_1^1, \dots, h_{n_1}^1, \dots, h_1^p, \dots, h_{n_p}^p\},$$

where h_j^i , $i = \overline{1, p}$, $j = \overline{1, n_i}$ are of the form (18).

3. A Fundamental Set of Solutions for the Equations (1) Based on a Jordan Basis in $\mathbb{C}_n$ of the Matrix A . Let be the spaces $S(\mathbb{R}, \mathbb{C})$ (the n -dimensional linear space of solutions of equation (1), defined on $\mathbb{R}$ with values in $\mathbb{C}$) and $T(\mathbb{R}, \mathbb{C}_n)$ the n -dimensional linear space of the vector-solutions of the system (2)). If the vector-function $x(t) = (x_1(t) \dots x_n(t))^T$, $t \in \mathbb{R}$ belongs to $T(\mathbb{R}, \mathbb{C}_n)$, we consider the function $\varphi(x(t)) = x_1(t)$, $t \in \mathbb{R}$. It is possible to verify that $\varphi(x(t))$, $t \in \mathbb{R}$, belongs to $S(\mathbb{R}, \mathbb{C})$ [1], [8]. After that we can define the application

$$\varphi: T(\mathbb{R}, \mathbb{C}_n) \rightarrow S(\mathbb{R}, \mathbb{C}), \quad \varphi(x) = x_1, \quad \forall x = (x_1 \ x_2 \dots x_n)^T \in T(\mathbb{R}, \mathbb{C}_n).$$

It is also proved that this application is an isomorphism. As a consequence of this fact, it results that, if the vector-system

$$\{x_1(t), \dots, x_n(t)\}, \quad t \in \mathbb{R}, \quad x_k(t) = (x_{k1}(t) \ \dots \ x_{kn}(t))^T, \quad t \in \mathbb{R},$$

belonging to $T(\mathbb{R}, \mathbb{C}_n)$, $k = \overline{1, n}$, is a basis of the space $T(\mathbb{R}, \mathbb{C}_n)$, then the function-system $\{x_{11}(t), x_{21}(t), \dots, x_{n1}(t)\}$, $t \in \mathbb{R}$, is a basis in the space $S(\mathbb{R}, \mathbb{C})$. So, having a basis in $T(\mathbb{R}, \mathbb{C}_n)$ we can write immediately a basis in $S(\mathbb{R}, \mathbb{C})$ [4].

A basis in $T(\mathbb{R}, \mathbb{C}_n)$ can be obtained in the following way: we consider in $\mathbb{C}_n$ a Jordan basis corresponding to the matrix A , of the form (19). Then, the vector system

$$(20) \quad \left\{ e^{\lambda_1 t} h_1^1, e^{\lambda_1 t} \left[\frac{t}{1!} h_1^1 + h_2^1 \right], \dots, e^{\lambda_1 t} \left[\frac{t^{n_1-1}}{(n_1-1)!} h_1^1 + \frac{t^{n_1-2}}{(n_1-2)!} h_2^1 + \dots + \frac{t}{1!} h_{n_1-1}^1 + h_{n_1}^1 \right], \dots \right. \\ \left. \dots, e^{\lambda_p t} \left[\frac{t^{n_p-1}}{(n_p-1)!} h_1^p + \frac{t^{n_p-2}}{(n_p-2)!} h_2^p + \frac{t}{1!} h_{n_p-1}^p + h_{n_p}^p \right] \right\}, \quad t \in \mathbb{R},$$

is a basis in $T(\mathbb{R}, \mathbb{C}_n)$ [8], [6].

Taking into account the form of the vectors h_j^i , $i = \overline{1, p}$, $j = \overline{1, n_i}$, it results that

the function-system

$$(21) \left\{ e^{\lambda_1 t}, \frac{t}{1!} e^{\lambda_1 t}, \dots, \frac{t^{n_1-1}}{(n_1-1)!} e^{\lambda_1 t}, \dots, e^{\lambda_p t}, \frac{t}{1!} e^{\lambda_p t}, \dots, \frac{t^{n_p-1}}{(n_p-1)!} e^{\lambda_p t} \right\}, t \in \mathbb{R},$$

is a basis in $S(\mathbb{R}, \mathbb{C})$, and obviously, the function system

$$(22) \left\{ e^{\lambda_1 t}, t e^{\lambda_1 t}, \dots, t^{n_1-1} e^{\lambda_1 t}, \dots, e^{\lambda_p t}, t e^{\lambda_p t}, \dots, t^{n_p-1} e^{\lambda_p t} \right\}, t \in \mathbb{R},$$

as well. So, because we have

$$(23) \det(\lambda I_n - A) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n,$$

it is proved the following well known

Theorem 2. Let $\lambda_1, \dots, \lambda_p$, $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = \overline{1, p}$, be the roots of the equation

$$(24) \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0,$$

(the characteristic equation of the differential equation (1)), with the multiplicities $n_1, \dots, n_p$, respectively.

Then, the function-system (22) is a basis in $S(\mathbb{R}, \mathbb{C})$.

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Technical University "Gh. Asachi", Jassy,
Department of Mathematics

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**BAZE JORDAN ȘI ECUAȚII DIFERENȚIALE LINIARE
DE ORDIN SUPERIOR, CU COEFICIENȚI CONSTANȚI**

(Rezumat)

Se consideră ecuația diferențială liniară de ordinul n ($n > 1$) (1) și sistemul de ecuații diferențiale liniare (2), ambele cu coeficienții constanți. Se determină bazele Jordan în C_n , corespunzătoare matricei A a sistemului (2) (Teorema 1) și o bază în $T(\mathbb{R}, C_n)$ (spațiul vectorial al soluțiilor sistemului (2), definite pe $\mathbb{R}$ cu valori în C_n) folosind baza (19). Utilizând un izomorfism între $T(\mathbb{R}, C_n)$ și $S(\mathbb{R}, C)$ (spațiul vectorial al soluțiilor ecuației (1), definite pe $\mathbb{R}$ cu valori în C) și baza găsită în $T(\mathbb{R}, C_n)$ se determină apoi o binecunoscută bază în $S(\mathbb{R}, C)$ (Teorema 2).

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(Continued)

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THE FINITE ELEMENT METHODS FOR A SINGULAR BOUNDARY VALUE PROBLEM

BY

MARIA BECEANU and ARIADNA-LUCIA PLETEA

Abstract. The boundary value problem (P) is considered, the differential operator being expressed by (1.1). The exact solution is approximated by a finite linear element method and an error estimation is performed.

Key words: boundary value problem, finite element method, weighted Sobolev space, piecewise linear interpolation

1. Statement of Problem. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a continuous Lipschitz boundary Γ .

We consider the differential operator

$$(1.1) \quad Au = - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_j} \right] - \frac{\partial}{\partial x_n} \left[a_{nn}(x) \frac{\partial u}{\partial x_n} \right] - \frac{2\mu}{x_n} a_{nn}(x) \frac{\partial u}{\partial x_n}, \quad \mu > 0.$$

We assume the conditions:

- $c_1) \quad a_{ij}(x) \in C^1(\bar{\Omega}), \quad \forall i, j = \overline{1, n};$
- $c_2) \quad a_{ij}(x) = a_{ji}(x), \quad \forall x \in \bar{\Omega} \quad \text{and} \quad i, j = \overline{1, n-1};$
- $c_3) \quad \text{the form } \sum_{i,j=1}^{n-1} a_{ij}(x) t_i t_j \text{ is positive definite;}$
- $c_4) \quad \text{the coefficient } a_{nn}(x) \text{ is a positive function } a_{nn}(x) \geq c > 0, \quad \forall x \in \Omega;$
- $c_5) \quad \text{the domain } \Omega \text{ is lying in the upper half-space } x_n > 0 \text{ and its boundary intersects the hyperplane } x_n = 0 \text{ in a hypersurface } \Gamma_0. \text{ The rest of } \partial\Omega \text{ is } \Gamma_1 \text{ situated in } x_n > 0.$

Under these conditions the boundary value problem

$$(P) \quad \begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial x_n} = 0 & \text{on } \Gamma_0, \end{cases}$$

is considered.

The behaviour of the finite element method in solving the problem (P) is analyzed.

2. Weighted Sobolev Spaces. Let μ be a positive real number. For $u \in C^\infty(\bar{\Omega})$ we define the weighted L_μ^2 norms

$$(2.1) \quad \|u\|_{L_\mu^2(\Omega)} = \left(\int_\Omega x_n^{2\mu} u^2(x) dx \right)^{1/2},$$

seminorms

$$(2.2) \quad |u|_{H_\mu^k(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|_{L_\mu^2(\Omega)}$$

and norms

$$(2.3) \quad \|u\|_{H_\mu^m(\Omega)} = \sum_{k=0}^m |u|_{H_\mu^k(\Omega)}.$$

Let $H_\mu^m(\Omega)$ be the completion of $C^\infty(\bar{\Omega})$ functions in the $H_\mu^m(\Omega)$ norm. Note that for $L_\mu^2(\Omega)$ the function $x_n^\mu u(x)$ is in $L^2(\Omega)$ since it is a limit of continuous functions in $L^2(\Omega)$; thus members of $L_\mu^2(\Omega)$ are measurable functions.

If we define $C_1^\infty(\Omega) = \{\varphi \in C^\infty(\bar{\Omega}) : \varphi|_{\Gamma_1} = 0\}$ then $\dot{H}_\mu^m(\Omega)$ is the completion of $C_1^\infty(\Omega)$ in the $H_\mu^m(\Omega)$ norm.

3. Variational Formulation. We define the bilinear form $a : H_\mu^1(\Omega) \times H_\mu^1(\Omega) \rightarrow \mathbb{R}$ and the linear functional $F : H_\mu^1(\Omega) \rightarrow \mathbb{R}$ as follows:

$$(3.1) \quad a(u, v) = \int_\Omega x_n^{2\mu} \left[\sum_{i,j=1}^{n-1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_{nn}(x) \frac{\partial u}{\partial x_n} \frac{\partial v}{\partial x_n} \right] dx$$

and

$$(3.2) \quad F(v) = \int_\Omega x_n^{2\mu} f(x) v(x) dx, \quad f \in L^2(\Omega).$$

The weight factor x_n^μ which degenerates at the $x_n = 0$ distinguishes this form from those that arise in elliptic problems where the corresponding weight would be bound away from zero.

Then we have the variational formulation of problem (P):

$$\text{find } u \in \dot{H}_\mu^1(\Omega) \text{ such that } a(u, v) = F(v) \text{ for all } v \in \dot{H}_\mu^1(\Omega).$$

First we shall prove a weighted Poincaré lemma. We assume that $u(x) \equiv 0$ for $x \notin (\bar{\Omega})$ and that Ω lies between two parallel hyperplanes. Without loss of generality, let Ω be between $x_n = 0$ and $x_n = l_n$.

L e m m a 1. If $u \in \dot{H}_\mu^1(\Omega)$ then $\|u\|_{L_\mu^2(\Omega)} \leq C|u|_{\dot{H}_\mu^1(\Omega)}$.

P r o o f. We enclose Ω in some parallelepiped Π and we assume, without loss of generality, that $\Pi = \{x/0 \leq x_i \leq l_i, i = \overline{1, n}\}$. Let be $u \in \dot{H}_\mu^1(\Omega)$ such that $u(x) = 0$ for $x \notin \Omega$. Then

$$\int_{\Pi} x_n^{2\mu} u^2(x) dx = - \int_{\Pi} \frac{x_n^{2\mu+1}}{2\mu+1} 2u \frac{\partial u}{\partial x_n} dx.$$

If we use Cauchy's inequality we obtain

$$\int_{\Pi} x_n^{2\mu} u^2(x) dx \leq \text{const.} \left[\int_{\Pi} (x_n^\mu u)^2 dx \right]^{\frac{1}{2}} \left[\int_{\Pi} \left(x_n^\mu \frac{\partial u}{\partial x_n} \right)^2 dx \right]^{\frac{1}{2}}.$$

From this it follows that $\|u\|_{L_\mu^2(\Omega)} \leq \text{const.} |u|_{\dot{H}_\mu^1(\Omega)}$.

The elliptic problem we use is as follows. For $\varepsilon > 0$ we define

$$\Omega_\varepsilon = \{(x_1, \dots, x_n) \in \Omega, x_n > \varepsilon\}, \quad \Gamma_{0\varepsilon} = \{(x_1, \dots, x_n) \in \partial\Omega_\varepsilon, x_n = \varepsilon\}, \quad \Gamma_{1\varepsilon} = \partial\Omega_\varepsilon - \Gamma_{0\varepsilon}.$$

Also let $\{f_k\} \subset C_0^\infty(\bar{\Omega})$ be a sequence that converges to f in $L_\mu^2(\Omega)$. We consider the problem

$$(P_\varepsilon) \quad \begin{cases} Au_{\varepsilon,k} = f_k & \text{in } \Omega_\varepsilon, \\ u_{\varepsilon,k} = 0 & \text{on } \Gamma_{1,\varepsilon}, \\ \frac{\partial u_{\varepsilon,k}}{\partial \nu} = 0 & \text{on } \Gamma_{0,\varepsilon}. \end{cases}$$

T h e o r e m 1. The problem (P_ε) has a unique weak solution $u_{\varepsilon,k} \in \dot{H}_\mu^1(\Omega_\varepsilon)$.

P r o o f. It follows from Lemma 1 and the assumptions $c_1) - c_5)$ that the bilinear form

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} x_n^{2\mu} \left[\sum_{i,j=1}^{n-1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_{nn}(x) \frac{\partial u}{\partial x_n} \frac{\partial v}{\partial x_n} \right] dx,$$

is coercitive and continuous on $H_\mu^1(\Omega_\varepsilon) \times H_\mu^1(\Omega_\varepsilon)$ and the linear functional $F_\varepsilon(v) = \int_{\Omega_\varepsilon} x_n^{2\mu} f_k(x) v(x) dx$ is continuous on $H_\mu^1(\Omega_\varepsilon)$. Hence the conclusion of the theorem is a

result of the Lax-Milgram theorem. The next theorem is a extension of the work of B e n d a l l i [1].

T h e o r e m 2. Let $\{f_k\} \subset C_0^\infty(\bar{\Omega})$ be a sequence that converges to f in $L_\mu^2(\Omega)$ norm and $u_{\varepsilon,k}$ the solution of problem (P_ε) . Then $u_{\varepsilon,k} \in H_\mu^2$ and there exists a

constant C such that

$$(3.3) \quad \|u_{\varepsilon,k}\|_{H_{\mu}^2(\Omega_{\varepsilon})} + \left\| \frac{1}{x_n} \frac{\partial u_{\varepsilon,k}}{\partial x_n} \right\|_{L_{\mu}^2(\Omega_{\varepsilon})} \leq C \|f_k\|_{L_{\mu}^2(\Omega_{\varepsilon})}.$$

P r o o f. The first part of this theorem results using local maps and partition of unity. We may take a finite covering of Ω_{ε} by open sets $\{K_i\}_{i=1}^N$ whose diameter will be sufficiently small and we associated partition of unity $\{\varphi_i\}_{i=1}^N$, $\varphi_i \in C_0^{\infty}(\mathbb{R}^n)$, the support of φ_i in K_i and $\sum_{i=1}^N \varphi_i = 1$ in Ω_{ε} . We have $u_{\varepsilon,k} = \sum_{i=1}^N \varphi_i u_{\varepsilon,k}$; therefore it suffices to consider $\varphi_i u_{\varepsilon,k}$ for each i .

By a similar argument to the one in B e n d a l l i [1] we then show that the solution $u_{\varepsilon,k}$ of problem (P _{ε}) is an element of $H_{\mu}^2(\Omega_{\varepsilon})$. Let

$$A u_{\varepsilon,k} = A_1 u_{\varepsilon,k} - \frac{2\mu}{x_n} a_{nn}(x) \frac{\partial u_{\varepsilon,k}}{\partial x_n},$$

where

$$A_1 u_{\varepsilon,k} = - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u_{\varepsilon,k}}{\partial x_j} \right] - \frac{\partial}{\partial x_n} \left[a_{nn}(x) \frac{\partial u_{\varepsilon,k}}{\partial x_n} \right].$$

Multiply both sides of $x_n^{2\mu} A_1 u_{\varepsilon,k}$ and integrate to obtain

$$\int_{\Omega_{\varepsilon}} x_n^{2\mu} (A_1 u_{\varepsilon,k})^2 dx - 2\mu \int_{\Omega_{\varepsilon}} x_n^{2\mu} A_1 u_{\varepsilon,k} \frac{1}{x_n} a_{nn}(x) \frac{\partial u_{\varepsilon,k}}{\partial x_n} dx = \int_{\Omega_{\varepsilon}} (A u_{\varepsilon,k} \cdot A_1 u_{\varepsilon,k}) x_n^{2\mu} dx.$$

Using the ellipticity condition for $A_1 u_{\varepsilon,k}$ and the Cauchy's inequality it follows

$$C_1 \|u_{\varepsilon,k}\|_{H_{\mu}^2(\Omega_{\varepsilon})} \leq \|A_1 u_{\varepsilon,k}\|_{L_{\mu}^2(\Omega_{\varepsilon})} \leq C_2 \|A u_{\varepsilon,k}\|_{L_{\mu}^2(\Omega_{\varepsilon})} + \left\| \frac{1}{x_n} \frac{\partial u_{\varepsilon,k}}{\partial x_n} \right\|_{L_{\mu}^2(\Omega_{\varepsilon})}.$$

But

$$\left\| \frac{1}{x_n} \frac{\partial u_{\varepsilon,k}}{\partial x_n} \right\|_{L_{\mu}^2(\Omega_{\varepsilon})} \leq C \|f_k\|_{L_{\mu}^2(\Omega_{\varepsilon})},$$

so

$$\|u_{\varepsilon,k}\|_{H_{\mu}^2(\Omega_{\varepsilon})} \leq \|u_{\varepsilon,k}\|_{H_{\mu}^2(\Omega_{\varepsilon})} + \left\| \frac{1}{x_n} \frac{\partial u_{\varepsilon,k}}{\partial x_n} \right\|_{L_{\mu}^2(\Omega_{\varepsilon})} \leq \text{const.} \|f_k\|_{L_{\mu}^2(\Omega_{\varepsilon})}.$$

Remark. Now we extend each $u_{\varepsilon,k}$ in $H_{\mu}^2(\Omega)$ by an even reflection from Ω_{ε} to Ω . So the sequence $(u_{\varepsilon,k})$ is bounded uniformly in $H_{\mu}^2(\Omega)$ and, according to the weak compactness of the unit ball in Hilbert spaces there exists a subsequence which converges weakly in $H_{\mu}^2(\Omega)$ to limit u_k . Using a Rellich's Lemma in $H_{\mu}^2(\Omega)$ we prove

that u_k solves problem (P) which right hand f_k and satisfies the inequalities

$$\|u_\varepsilon\|_{H_\mu^2(\Omega_\varepsilon)} \leq C \|f_k\|_{H_\mu^2(\Omega)}, \quad \left\| \frac{1}{x_n} \frac{\partial u_k}{\partial x_n} \right\|_{L_\mu^2(\Omega_\varepsilon)} \leq C \|f_k\|_{H_\mu^2(\Omega)}.$$

Let u be a weak limit of subsequence of $\{u_k\}$. Then $u \in H_\mu^2(\Omega)$ and

$$\|u\|_{H_\mu^2(\Omega)} + \left\| \frac{1}{x_n} \frac{\partial u}{\partial x_n} \right\|_{L_\mu^2(\Omega)} \leq C \|f\|_{L_\mu^2(\Omega)}.$$

4. The Numerical Approximation. For the numerical solution of the problem (P) we suppose that Ω is a polyedric domain. Let $T_h = \{\Sigma_1, \dots, \Sigma_N\}$ be a triangulation of Ω , Σ_i a generic n -dimensional simplex of T_h . Let h_i be the maximum edge of the triangle Σ_i , $h = \max_i h_i$, $\theta = \min_i \theta_i$ and

$$(a) \quad \theta_i \geq \theta_0 > 0, \quad \theta_0 \text{ independent of } h.$$

We suppose that all triangulations coincide exactly with Ω and in the theoretical convergence estimate we assume that the integrals are computed exactly.

Define a linear space S_h as follows:

$$S_h = \{v_h \in C^0(\bar{\Omega}), v_h \text{ is linear function in } \Sigma_i, i = \overline{1, N}; v_h|_{\Gamma_1} = 0\}.$$

It is easy to observe that $S_h \subset H_\mu^2(\Omega)$.

Based on problem (P) we use the following finite element formulation:

(P_h) find $u_h \in S_h$ such that $a(u_h, v_h) = F(v_h)$ for all $v_h \in S_h$.

This results in a linear system of equations; since the bilinear form is positive definite, there will exists a unique solution. Also, since $S_h \subset H_\mu^1(D)$, we have $a(u - u_h, v_h) = 0$ for all $v_h \in S_h$, where u is the solution of problem (P). Thus u_h is the $H_\mu^1(D)$, the projection of u into S_h . Using Cea's Lemma we have

$$|u - u_h|_{H_\mu^1(D)} \leq \inf_{v_h \in S_h} |u - v_h|_{H_\mu^1(D)}.$$

Assume that u is the solution of problem (P), u_I the piecewise linear interpolation corresponding to the triangulation T_h . For any simplex $\Sigma \in T_h$ we now estimate $\|u - u_I\|_{H_\mu^1(\Sigma)}$. Let $(P_i)_{i=\overline{1, n+1}}$ the vertex of Σ , $\lambda_i(P)$ the so called barycentric coordinates, i.e. the basis for the linear interpolation on Σ , $\lambda_j(P_i) = \delta_{ij}$. Then we have for any function u defined on Σ its linear interpolation u_I ,

$$u_I(P) = \sum_j \lambda_j(P) u(P_j).$$

L e m m a 2. We assume that $u \in C^1(\Sigma) \cap C^2(\text{int } \Sigma)$ and condition (a) is true.

Then

$$(4.1) \quad \|u - u_I\|_{H_\mu^1(\Sigma)} \leq Ch |u|_{H_\mu^2(\Sigma)}.$$

P r o o f. Because $u \in C^1(\Sigma) \cap C^2(\text{int } \Sigma)$ we can write the Lagrange polynomial for function so that

$$u(P) - u_I(P) = \frac{1}{2} \sum_{i=1}^{n+1} D^2(u)(\theta_i(P) + (1-\theta)P)(P_i - P)^2 \lambda_i(P).$$

Using the evaluations from [4 p. 121] and because Ω is polyedric domain we obtain

$$\|u - u_I\|_{L^2_\mu(\Sigma)} \leq C_1 h^2 |u|_{H^2_\mu(\Sigma)} \quad \text{and} \quad |u - u_I|_{H^1_\mu(\Sigma)} \leq C_2 h |u|_{H^2_\mu(\Sigma)}$$

so that we have (4.1).

L e m m a 3. Let $b_\mu u$ and u_I be as above. Then there exists a constant $C = C(\mu, \Omega)$ such that if $u \in H^2_\mu(\Omega)$ we have

$$\|u - u_I\|_{H^1_\mu(\Omega)} \leq Ch |u|_{H^2_\mu(\Omega)}.$$

P r o o f. By virtue of Lemma 2 we have

$$\|u - u_I\|_{H^1_\mu(\Omega)}^2 = \sum_{i=1}^N \|u - u_I\|_{H^1_\mu(\Sigma_i)}^2 \leq Ch^2 |u|_{H^2_\mu(\Omega)}^2.$$

C o r o l l a r y. We have

$$\|u - u_I\|_{H^1_\mu(\Omega)} = o(h), \quad \|u - u_I\|_{L^2_\mu(\Omega)} = o(h^2).$$

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Technical University "Gh. Asachi", Jassy,
Department of Mathematics

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METODA ELEMENTULUI FINIT PENTRU O PROBLEMĂ SINGULARĂ CU VALORI LA LIMITĂ

(Rezumat)

Se studiază o problemă cu valori la limită cu singularități pe frontieră, soluția exactă fiind aproximată cu elemente finite liniare. Se obține o evaluare a erorii în spații cu ponderi convenabil alese.

LES TYPES DE CIRCULATIONS ADMISSIBLES DANS UN GRAPHE ESCALIER NON ORIENTÉ (III)

PAR

VIOREL GHIUȘ

Abstract. The problem of admissible alternant circulation in a nonoriented scale graph is studied and a necessary and sufficient existence condition is established.

Key words: graph, nonoriented graph, alternant circulation.

Soit un graphe escalier [5] G avec $f = \overline{aa'}/a'a$ où les arêtes sont non orientées; les flèches du graphe indiquent le sens de la circulation qui sera introduite plus tard.

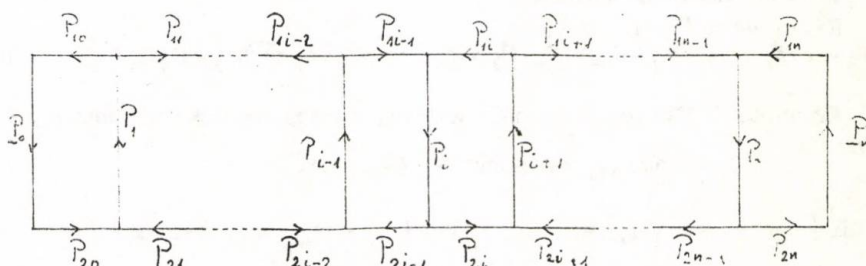


Fig. 1

Nous noterons les fonctions de circulation et de capacité:

$$F(P_i) = F_i, \quad c(P_i) = c_i, \quad C(P_i) = C_i, \quad i = \overline{1, n};$$

$$F(P_{ki}) = F_{ki}, \quad c(P_{ki}) = c_{ki}, \quad C(P_{ki}) = C_{ki}, \quad i = \overline{0, n} \text{ et } k = 1, 2;$$

$$F(P_h) = F_h, \quad c(P_h) = c_h, \quad C(P_h) = C_h, \quad h = 0, n.$$

Définition 1. On appelle une circulation dans un graphe escalier G [5], une paire (f, F) , où f est une orientation $\overline{aa'}/a'a$ ou $\overline{a'a}/aa'$ (équivalentes [5]) de G

(Fig. 1) et F une circulation dans G^f (le graphe orienté attaché), c'est à dire qui satisfait les équations de conservation

$$(1) \quad \begin{cases} F_i = F_{ki-1} + F_{ki}, & k=1,2 \text{ et } i=\overline{1,n}; \\ \underline{F}_h = F_{kh}, & k=1,2 \text{ et } h=\overline{0,n}. \end{cases}$$

Cette circulation est admissible si elle satisfait les conditions

$$(2) \quad \begin{cases} \underline{c}_h \leq \underline{F}_h \leq \underline{C}_h, & h=\overline{0,n}; \\ c_i \leq F_i \leq C_i, & i=\overline{1,n}; \\ c_{ki} \leq F_{ki} \leq C_{ki}, & k=1,2 \text{ et } i=\overline{0,n}. \end{cases}$$

Remarque 1. De (1) résulte:

a) le sens de la circulation sur les arêtes P_{ki-1} est opposé au sens de la circulation sur les arêtes P_{ki} , $k=1,2$ et $i=\overline{1,n}$;

b) le sens de la circulation sur les arêtes P_{1i} est opposé au sens de la circulation sur les arêtes P_{2i} , $i=\overline{0,n}$;

c) le sens de la circulation sur les arêtes $\underline{P}_0, P_2, \dots, P_i, \dots, P_n$ est opposé au sens de la circulation sur les arêtes $P_1, P_3, \dots, P_{i-1}, P_{i+1}, \dots, P_{n-1}, \underline{P}_n$.

Cette circulation est alternative.

Remarque 2. $F_{1i} = F_{2i}$, $i=\overline{0,n}$;

$$F_1 + F_3 + \dots + F_{i-1} + F_{i+1} + \dots + F_{n-1} + \underline{F}_n = \underline{F}_0 + F_2 + \dots + F_i + \dots + F_n = \sum (F_{ki}; 0 \leq i \leq n), \quad k=1,2.$$

Remarque 3. Des conditions précédentes combinées avec les conditions (2) il résulte:

$$\max(c_{1i}, c_{2i}) \leq \min(C_{1i}, C_{2i}), \quad i=\overline{0,n};$$

$$\begin{aligned} \max(c_1 + c_3 + \dots + c_{i-1} + c_{i+1} + \dots + \underline{c}_n, \sum (c_{ki}; 0 \leq i \leq n), \underline{c}_0 + c_2 + \dots + c_i + \dots + c_n) \leq \\ \leq \min(C_1 + C_3 + \dots + C_{i-1} + C_{i+1} + \dots + C_{n-1} + \underline{C}_n, \underline{C}_0 + C_2 + \dots + C_i + \\ + \dots + C_n, \sum (C_{ki}; 0 \leq i \leq n), \quad k=1,2. \end{aligned}$$

On définit les quantités:

$$\dot{c}_i = \max(c_{1i}, c_{2i}), \quad \dot{C}_i = \min(C_{1i}, C_{2i}), \quad i=\overline{0,n};$$

$$c_0'' = \max(\underline{c}_0, \dot{c}_0), \quad \underline{C}_0'' = \min(\underline{C}_0, \dot{C}_0);$$

$$c_1'' = \max(c_0'' + \dot{c}_1, c_1) = c_0' + \bar{c}_1, \quad C_1'' = \min(C_0'' + \dot{C}_1, C_1) = C_0' + \bar{C}_1$$

où

$$c_0' \geq c_0'', \bar{c}_1 \geq \dot{c}_1, c_0' \leq c_0'', \bar{c}_1 \leq \dot{c}_1.$$

Généralement

$$c_{i+1}'' = \max(\bar{c}_i + \dot{c}_{i+1}, c_{i+1}') = c_i' + \bar{c}_{i+1}, c_i' \geq \bar{c}_i, \bar{c}_{i+1} \geq \dot{c}_{i+1},$$

$$C_{i+1}'' = \min(\bar{C}_i + \dot{C}_{i+1}, C_{i+1}') = C_i' + \bar{C}_{i+1}, C_i' \leq \bar{C}_i, \bar{C}_{i+1} \leq \dot{C}_{i+1}, i = \overline{1, n-1}$$

et enfin

$$c_{n+1}'' = \max(\bar{c}_n, \underline{c}_n), C_{n+1}'' = \min(\bar{C}_n, \underline{C}_n).$$

Remarque 4. Nous avons:

$$c_i \leq c_i'' \text{ et } C_i'' \leq C_i, i = \overline{1, n};$$

$$c_0 \leq c_0'', c_0'' \leq \underline{c}_0, \underline{c}_n \leq c_{n+1}'', C_{n+1}'' \leq \underline{C}_n,$$

$$c_{ki} \leq \dot{c}_i \leq \bar{c}_i \leq c_i', C_{ki} \geq \dot{C}_i \geq \bar{C}_i \geq C_i', i = \overline{1, n}, k = 1, 2.$$

Remarque 5. Généralement les conditions

$$(3) \quad \begin{cases} c_i'' \leq C_i'', & i = \overline{0, n+1}, \\ c_i' \leq C_i', & i = \overline{0, n}, \end{cases}$$

ne sont pas satisfaites.

T h é o r è m e. La condition nécessaire et suffisante que dans un graphe escalier $\mathcal{G}$ (Fig. 1) existe une circulation (alternative) admissible (f, F) , où f est une orientation $\overline{aa'}/aa'$ ou $\overline{a'a}/a'a$ et F une circulation de la forme (1) en satisfaisant (2), est que la condition (3) soit satisfaite.

N é c e s s i t é. Des conditions (1) et (2) nous avons:

$$\left. \begin{aligned} F_h &= F_{kh}, & k=1, 2, \quad h=0, n; \\ c_{kh} &\leq F_{kh} \leq C_{kh}, & k=1, 2, \quad h=0, n; \\ \underline{c}_h &\leq \underline{F}_h \leq \underline{C}_h, & h=0, n, \end{aligned} \right\} \Rightarrow c_h'' \leq F_h \leq C_h'', h=0, n$$

en supposant maintenant

$$\begin{cases} c_{i-1}' \leq F_{ki-1} \leq C_{i-1}', & k=1, 2, \\ \bar{c}_i \leq F_{ki} \leq \bar{C}_i & k=1, 2, \end{cases}$$

combiné avec

$$\begin{cases} F_{i+1} = F_{ki} + F_{ki+1}, & k=1, 2, \\ c_{i+1} \leq F_{i+1} \leq C_{i+1}, & k=1, 2, \end{cases}$$

il résulte

$$\begin{cases} c_{i+1}'' \leq F_{i+1} \leq C_{i+1}'', \\ c_i' \leq F_{ki} \leq C_{ki}', \\ \bar{c}_{i+1} \leq F_{ki+1} \leq \bar{C}_{i+1}, \end{cases} \quad k=1,2.$$

Suffisance. On définit

$$F_{ki} = s_i \quad \text{avec} \quad c_i' \leq n_i \leq C_i' \quad \text{pour} \quad k=1,2 \quad \text{et} \quad i=\overline{0,n};$$

$$F_i = s_{i-1} + s_i \quad \text{pour} \quad i=\overline{1,n};$$

$$F_h = F_{kh} = s_h \quad \text{pour} \quad h=0,n.$$

Les équations de conservation (1) sont évidemment satisfaites. On vérifie que les conditions (3) sont satisfaites aussi; en effet

$$c_i \leq c_i'' = c_{i-1}' + \bar{c}_i \leq c_{i-1}' + c_i' \leq s_{i-1} + s_i = F_i \leq C_{i-1}' + C_i' \leq C_{i-1}' + \bar{C}_i = C_i'' \leq C_i,$$

donc

$$c_i \leq F_i \leq C_i, \quad i=\overline{1,n}.$$

Continuant ainsi, on obtient

$$c_{ki} \leq \dot{c}_i \leq \bar{c}_i \leq c_i' \leq s_i = F_{ki} \leq C_i' \leq \bar{C}_i \leq \dot{C}_i \leq C_{ki}, \quad k=1,2,$$

donc

$$c_{ki} \leq F_{ki} \leq C_{ki}, \quad k=1,2 \quad \text{et} \quad i=\overline{1,n}.$$

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Chaire de Mathématiques

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TIPURI DE CIRCULAȚII ADMISIBILE ÎNTR-UN GRAF SCARĂ NEORIENTAT (III)

(Rezumat)

Se studiază problema circulației alternante [5] (de forma $\overline{aa'}/a'a$ sau $\overline{a'}/aa'$) admisibilă dintr-un graf scară neorientat și se găsește o condiție necesară și suficientă de existență.

HIDDEN MARKOV MODELS IN AUTOMATIC SPEECH RECOGNITION
A SPEAKER-INDEPENDENT APPLICATION TO A
VOCABULARY OF SPANISH WORDS

BY

JESÚS LÓPEZ-FIDALGO and GUSTAVO-SANTOS-GARCÍA

Abstract. A hybrid system of automatic speech recognition for isolated Spanish words, independent of the speaker, is developed using Hidden Markov Models (HMM) and Vector Quantization (VQ). An exhaustive review of the working and operability of the HMM is carried out. The mathematical environment MatLab for the computer development has been used. In the system proposed was performed a training stage for the models with sound samples coming from several speakers (half of which were men and the other half women). For testing the models another group of speakers different from those of the training stage was used. Finally, the results obtained with this recognition method are given.

Key words: automatic speech recognition, hidden Markov models, vector quantization.

1. Introduction. Speech recognition is a very complex problem. The complex and variable way in which language messages are codified makes it very difficult. Many government-funded speech research projects for the military sector have bolstered the growth of the speech industry as a whole. Technical improvements made under military auspices are being applied elsewhere in a host of commercial applications.

In the field of speech a lot of studies have been carried out by many researchers for many years now. These researchers have been encouraged by the huge development of computers in speed, storage, interfaces, etc. In this area we can find several applications that are very different in foundation and performance: storage and digital transmission of the speech signal, synthesis of speech, speaker identification and verification, speech recognition, handicappers aid, improvement of signal quality, etc.

Techniques of digital signal processing are increasing quickly in speech analysis. These techniques, which can be applied easily on a normal computer, make it possible to design and perform techniques of Automatic Speech Recognition (ASR).

Looking at the shape of the vocal tract, which basically and simply is a cylindrical pipe closed on one side by the vocal cords and opened on the other side by the lips, allows us to set up a model of speech production based on principles of digital signal processing. A simplified picture of the human vocal tract is represented in Fig. 1.

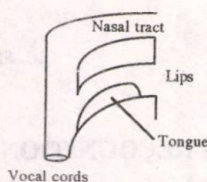


Fig. 1 — Scheme of human vocal tract.

In speech recognition there are two great fields [6]: isolated speech recognition and continuous speech recognition. This study belongs to the first one and is made with independence of the speaker. A discrete set of words or phonemes forms the vocabulary of the recognizer. The recognizing system is shown in Fig. 2. Finally a sound is given to the recognizer and it will assign an element of the vocabulary to this sound.

In Continuous Speech Recognizing (CSR), a sentence or text is pronounced. Therefore, there is the added problem of finding where the word starts and finishes. It will be necessary to determine the silence and also the problem of recognizing every word as before.

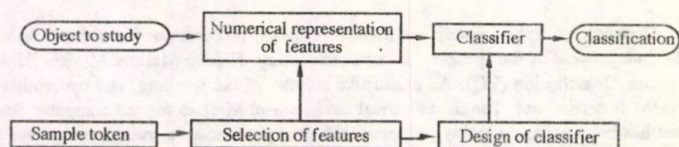


Fig. 2 — Scheme of recognizing system.

There are many techniques for speech recognition. Some of them are: Hidden Markov Models [4], [5], [10], Pattern Recognition, Statistics Recognizers [12], Neural Networks [7], [13], etc. These general purpose recognizers take speech data through the speech signal. The sources of speech yield and the shape of the vocal tract are relatively independent [9]. Most hybrid models for continuous speech recognition or isolated word recognition have fixed architecture during training, but it is possible to implement a self-structuring hidden control neural model for isolated word recognition [14]. A powerful system is the SRI-DECIPHER system which is a phone-based, speaker-independent, continuous-speech recognition system, based on semicontinuous HMMs [2] developed at SRI International.

From these data it is possible to obtain, through digital signal processing techniques, either a special codification or a parameter series. These can be divided in a temporal and spectral method depending on the methods used to obtain them (fundamental frequency, formant frequencies, energy per band, etc.).

2. Hidden Markov Models. We will use Hidden Markov Models as classifiers. A HMM needs a primary training stage to work properly. In this stage the initial parameters of the model are fitted to the real data. Later on the model can be used directly and system efficiency is then evaluated [11]. This process is shown in Figs. 3 and 4.

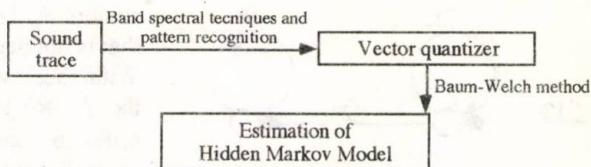


Fig. 3 — Scheme of hidden Markov model training.

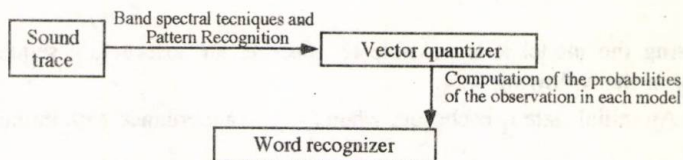


Fig. 4 — Scheme of hidden Markov model test.

The number of states of the model will be denoted by N . The definition of state depends on the specific problem to study, but every definition has one thing in common, which is that the signal in a state has something measurable, i.e., certain distinguishable properties. In speech recognition the states correspond to a temporal sequence of situations, which we can associate experimentally with temporal parts of the sound.

At each time t the process will come into a state, which is based on a distribution of transition probabilities depending on previous state (property of markovian processes). It is interesting to remark that transition may be such that the process remains at the previous state.

When transition has been performed, observation of the symbol shall be carried out according to a probability distribution depending only on the actual state. This probability distribution will be fixed for every state independently of the moment and the way in which it has arrived at that state. Therefore we will have N observation probability distributions representing random variables or stochastic processes.

T will denote the observation sequence length, that is, the number of observations in every trace analyzed. M will be the possible symbols that can be observed. Therefore we have a set of states that will be denoted by $Q = \{q_1, q_2, \dots, q_N\}$. The discrete set of possible symbols will be $V = \{v_1, v_2, \dots, v_M\}$. The $N \times N$ transition matrix between states will be $A = \{a_{ij}\}$, where $a_{ij} = P(q_j \text{ at time } t+1 \mid q_i \text{ at time } t)$ is the probability of reaching state q_j from state q_i , that is, not depending on t since it is a markovian process. In Fig. 5 we have a scheme of a very simple case.

The probability distribution of observable symbols at state j will be carried out by the M dimensional vector $B_j = \{b_j(k)\}$, where $b_j(k) = P(v_k \text{ at } t \mid q_j \text{ at } t)$ is the

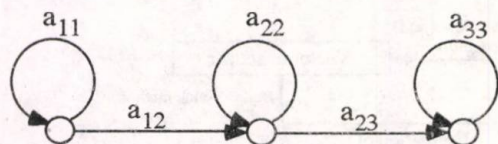


Fig. 5 — Acoustic model topology.

probability of choosing the symbol v_k being at state q_j , that is not depending on time. With these vectors we form the $N \times M$ matrix B . In order to complete the model we must give an initial distribution of the states, that will

be denoted, as in the general theory of Markov Chains, by the N dimensional vector $\pi = \{\pi_i\}$, where $\pi_i = P(q_i \text{ at time } t=1)$ is the probability of starting the process at state q_i .

Using the model it is possible to generate an observation sequence $O = O_1, O_2, \dots, O_T$ in the following way:

1. An initial state i_1 is chosen, when $t=1$, in accordance with initial distribution π ;
2. An observation O_t is chosen in accordance with the probability distribution of the symbols at state i_t , $b_{i_t}(k)$, for $k=1, \dots, M$;
3. A new state i_{t+1} is chosen from state i_t in accordance with the probability distribution of transition $\{a_{i_t i_{t+1}}\}$, where $i_{t+1} = 1, \dots, N$;
4. Making $t=t+1$ the process turns to 2 while $t < T$; otherwise the process is concluded.

From now on we will use notation $\lambda = (A, B, \pi)$ for a Hidden Markov Model (HMM). For the specification of a HMM it is necessary to choose the number N of states, the set of symbols V (in this work a discrete set of symbols with M elements will be used, although the non-discrete case could be studied also in a more general theory) and the three probability densities A , B and π . For most of the applications the π vector, which represents the initial conditions of the model, is the one of least importance. The most relevant distribution is, in general, matrix B since it is directly referred to the observed symbols. On the other hand we can say that the matrix A is important in certain problems, but not very important in others, such as the case considered in this paper — recognition of isolated words.

3. Computation of the Probability of an Observation for a Hidden Markov Model. The first problem is to obtain the probability of a certain observation sequence $O = O_1, O_2, \dots, O_T$ for a given model $\lambda = (A, B, \pi)$. The problem can be seen as the evaluation model for a specific observation. This point of view is very useful because we want to choose a model among several possible models in order to fit it better to a given observation. We will now compute this probability.

The shortest way to make this computation is to consider every possible sequence of states with length T . For each of these sequences $I = i_1 i_2 \dots i_T$, the probability of the observation sequence $O = O_1, O_2, \dots, O_T$ is the following:

$$P(O|I, \lambda) = b_{i_1}(O_1) \dots b_{i_T}(O_T).$$

On the other hand, the probability of each sequence of states is

$$P(I|\lambda) = \pi_{i_1} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{T-1} i_T},$$

and using the total probability rule,

$$\begin{aligned} P(I|\lambda) &= \sum_I P(O|I, \lambda) P(I|\lambda) = \\ &= \sum_{i_1, i_2, \dots, i_T} \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) a_{i_2 i_3} \dots a_{i_{T-1} i_T} b_{i_T}(O_T). \end{aligned}$$

The physical interpretation of the last formula is the following: fixing a sequence of states I , initially at time $t=1$, the process is at state i_1 with probability π_{i_1} and the symbol O_1 is generated with probability $b_{i_1}(O_1)$. Then, a transition to state i_2 shall be made with probability $a_{i_1 i_2}$, and the symbol O_2 shall be chosen with probability $b_{i_2}(O_2)$. The process continues in this way until the last transition from state i_{T-1} to state i_T with probability $a_{i_{T-1} i_T}$, where the symbol O_T is chosen with probability $b_{i_T}(O_T)$. The total probability is the result of summing these terms for every possible sequence of states.

This is very easy to verify, the computation of this probability needs $(2T-1)N^T$ multiplications, because every term of the sum has $2T-1$ factors and the total number of terms of the sum are the possible variations with repetition of T states among N . Moreover it is necessary to make N^T-1 sums. This computation could be an overflow, even for low values of N and T . For example, if $N=5$ and $T=100$ about $2 \cdot 100 \cdot 5^{100} \approx 10^{72}$ operations will be required. Therefore it is necessary to look for a more efficient process for the computation of this probability. For this purpose the following *forward-backward* process will be used.

Let us consider the *forward-backward* variable

$$\alpha_t(i) = P(O_1, O_2, \dots, O_t, i_t = q_i | \lambda),$$

that is, the probability of a partial observation sequence until time t and the state q_i at time t , for a given model λ . We can compute these probabilities recurrently as follows:

1. $\alpha_1(i) = \pi_i b_i(O_1)$, for $1 \leq i \leq N$;
2. For each $t=1, 2, \dots, T-1$ and $1 \leq j \leq N$ the recurrency law

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$

is true;

3. Then,

$$P(O|\lambda) = \sum_{i=1}^N \alpha_T(i).$$

In the first step the probability of state q_i appears and the initial observation O_1 , which has been calculated using the conditional probability definition. In the second

step the probability is computed according to the form of passing to state q_j at time $t+1$ from the possible states q_i , for $i=1, 2, \dots, N$ at time t . On the one hand $\alpha_t(i)$ is the probability of $O_1, O_2, \dots, O_t$ and state q_i at which the process remains at time t . Hence the product $\alpha_t(i)a_{ij}$ will be the probability of $O_1, O_2, \dots, O_t$ and state q_j at which the process remains at time $t+1$ assuming that the process was at state q_i at time t . Summing this product over the N possible states q_i , for $i=1, 2, \dots, N$, at time t , the probability of q_j at time $t+1$ is obtained with all previous observations. Thus, the probability $\alpha_{t+1}(j)$ will be the product of the last probability multiplied by $b_j(O_{t+1})$. Finally, in the third step the probability $P(O|\lambda)$ is obtained as the sum of the probabilities $\alpha_T(i)$, since, $\alpha_T(i)=P(O_1, O_2, \dots, O_T, i_T=q_i|\lambda)$.

The computation of each $\alpha_t(j)$, $1 \leq t \leq T$, $1 \leq j \leq N$, demands a number of operations of the order of N^2T , whereas the number of operations in the direct way were of the order of $2TN^2$. Precisely $N(N+1)(T-1)+N$ multiplications and $N(N-1)(T-1)$ sums. In the case of $N=5$ and $T=100$ only about 3000 operations will be necessary whereas 10^{72} operations are necessary in the direct way. This means a saving of 10^{69} operations with this method.

The key of this process is that there are only N states. Thus every sequence of states links these N states, the sequence length not being relevant. We have implemented this process with MatLab,

```
% Computation of alphas
% alpha t(i), 1 <= i <= N, 1 <= t <= T
alpha=zeros(T, N); % initialization of alpha.
alpha(1,:)=pi'.*B(0(1),:);
for t=1: T-1
    for j=1:N,
        alpha (t+1, j)=(alpha(t,:)*A(:,j))*B(0(t+1),j);
    end,
end,
```

First the alpha matrix is initialized and then the correspondent values for each element of this matrix are given in a recurrent way. Thus, the variables for $t=1$ are defined in the first step. Afterwards values for the remaining t are given through a loop. $O(t)$ represents the observation sequence taken of the observations performed for several speakers. This sequence is introduced in each step of the process and enables the model to be improved step by step. The matrix $A(i, j)$ is the transition matrix and $B(i, j)$ the probability of each symbol at each state. These matrices are introduced initially and at each step of the process they are modified. The possibility of using the matrix product facilitates the assignation of these variables. The probability of each observation sequence is,

```
% Computation of P(O|lambda)
PO=sum(alpha(T,:));
```

In a similar way we can now consider the following *backward* variable, $\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots, O_T | i_t = q_i, \lambda)$, that is, the probability of a partial sequence of observations from time $t+1$ to the end, given a state q_i at time t and in the model λ .

As before, we can compute the probabilities in a recurrent way:

1. $\beta_T(i) = 1, 1 \leq i \leq N$;
2. For $t = T - 1, T - 2, \dots, 1$ and $1 \leq i \leq N$ we have

$$\beta_t(i) = \sum_{j=1}^N a_{ij} b_j(O_{t+1}) \beta_{t+1}(j).$$

The computation of $\beta_t(i)$ for $1 \leq t \leq T, 1 \leq i \leq N$, needs — as in $\alpha_t(i)$ — a number of operations of the order of $N^2 T$.

The MatLab program we have developed for computing these variables is the following:

```
% Computation of betas
% beta t(i), 1 <= i <= N, 1 <= t <= T
beta=zeros (T, N); % initialization of beta
beta (T,:)=ones (1, N);
for t=T-1:-1:1,
    beta(t,:)=(B(O(t+1),:).'*beta(t+1,:))*A';
end,
```

where the initial values of the vector are introduced and then, with a loop, the elements of the variable beta are introduced. The matrices A and B are involved in this definition as in the case of alpha.

4. Fitting the Model with the Maximum Probability. Finally, we try to look for the parameters of the model $\lambda = (A, B, \pi)$ making the probability $P(O|\lambda)$ the maximum, i.e., a search will be made for the model best fitted to the observed sequence. For this reason we will say that it is a *training* sequence, because it is used for training the model. The training problem is very important for most of the applications of HMM, because it makes it possible to optimize the parameters of the model for the observed data, i.e., to create the best models for real phenomena. Nevertheless this problem has no easy solution. In fact, no analytic process for finding the model of maximum probability is known. For this reason it will be necessary to use algorithms like the Baum-Welch algorithm or gradient techniques. We will now discuss the iterative process, with which the physical meaning of estimation of the parameters can be seen.

We shall start with the following definition, $\xi_t(i, j) = P(i_t = q_i, i_{t+1} = q_j | O, \lambda)$, which is the probability of a path passing through state q_i at time t , and a transition of this state to state q_j occurs at time $t+1$, given an observation sequence for a specific model. Therefore, this probability can be written as follows,

$$\xi_t(i, j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{P(O|\lambda)},$$

where $\alpha_t(i)$ takes into account the first t observations, the term $a_{ij} b_j(O_{t+1})$ refers to transition to state q_j at time $t+1$ and the observation of the symbol O_{t+1} , while the term $\beta_{t+1}(j)$ makes reference to the rest of the observation sequence. $P(O|\lambda)$ is the normalization factor that makes the product exactly the probability $\xi_t(i, j)$.

We define $\gamma_t(i) = P(i_t = q_i | O, \lambda)$, which is the probability of the process being

at state q_i at time t , given the observation sequence O for a specific model. Thus it is possible to relate $\gamma_t(i)$ and $\xi_t(i, j)$ using the following sum of these probabilities over j , $\gamma_t(i) = \sum_{j=1}^N \xi_t(i, j)$.

The sum of $\gamma_t(i)$ over t can be interpreted as the expected value for the number of times the process passes through state q_i , or what is the same, the expected number of transitions made from state q_i , excluding the last time T , because from the last state there is no possible transition. Analogously, the sum of $\xi_t(i, j)$ from $t=1$ to $t=T-1$ can be understood as the expected number of transitions from state q_i to state q_j , i.e.,

$$\sum_{t=1}^{T-1} \gamma_t(i) = \text{expected number of transitions from } q_i,$$

$$\sum_{t=1}^{T-1} \xi_t(i, j) = \text{expected number of transitions from } q_i \text{ to } q_j.$$

With these formulae we can use the Baum-Welch method for the re-estimation of the parameters of HMM. The re-estimation formulae for π , A and B are the following:

1. $\bar{\pi}_i = \gamma_1(i), \quad 1 \leq i \leq N;$
2. $\bar{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i, j)}{\sum_{t=1}^{T-1} \gamma_t(i)}, \quad 1 \leq i, j \leq N;$
3. $\bar{b}_j(k) = \frac{\sum_{t=1; O_t=k}^{T-1} \gamma_t(j)}{\sum_{t=1}^{T-1} \gamma_t(j)}, \quad 1 \leq j \leq N, 1 \leq k \leq M.$

The estimation formula for π_i is easily the probability of being at state q_i at time $t=1$. The re-estimation formula for $\bar{a}_{ij}$ is the ratio between the expected number of transitions from state q_i to state q_j and the expected number of transitions from state q_i (for this reason the sum is made only until $T-1$). In the same way, the re-estimation of $\bar{b}_j(k)$ is the ratio between the expected number of times the process stays at state j and the symbol k is observed, and the expected number of times the process stays at state j . We have used, to some extent, the frequency concept of probability as a ratio between absolute frequency and total frequency.

If we take an initial model λ and the re-estimation $\bar{\lambda}$ of that model, then it is possible to say that, either the initial model λ defines a critical point of likelihood function, that is $\bar{\lambda} = \lambda$, or, the model $\bar{\lambda}$ is more likely in the sense that $P(O|\bar{\lambda}) > P(O|\lambda)$, i.e., we have found a model $\bar{\lambda}$ verifying that the observation sequence is yielded with greater probability.

Therefore, using $\bar{\lambda}$ instead of λ in an iterative way and repeating the last re-estimations we can improve the probability of O in the obtained model step by step until the process achieves a determined limit point. The final result is the estimated model we wish to find.

5. Conclusions. 1. Summarizing the work we have considered the following speech recognition scheme: we design a HMM with N states for every word of a given vocabulary V . Using techniques of Vectorial Quantization (VQ), we can show the speech signal as a sequence of symbols of a code VQ derived from a set of codes [10], [12]. In this way we start with a training sequence for each word of the vocabulary, that is a series of repetitions of the spoken word (by one or more speakers). We use the solutions calculated before for optimizing the parameters of the model for every word. For understanding the physical sense of states of the model we will use the solution given by the well-known Viterbi algorithm for segmenting the training sequence of each word into states, and will thus be able to study the observations in each state. The result of this study can be the improvement of the model. Finally, for recognizing a word, the solution provided by the computation of the probability of an observation for a given model will be used in order to *score* the model of every word from the observation sequence performed and the word with the greatest score will be chosen.

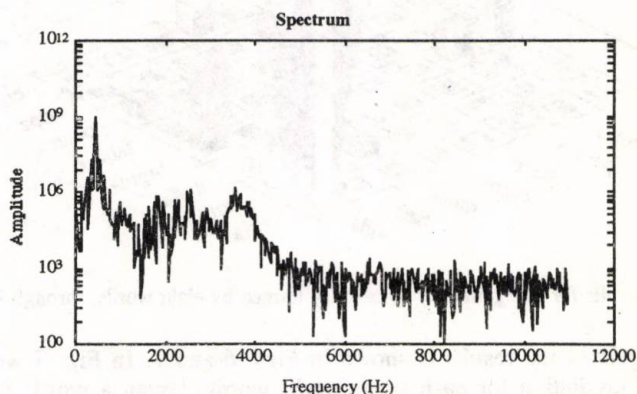


Fig. 6 — Spectrum of a segment of speech signal.

2. We have considered a vocabulary V of eight Spanish words. Specifically: $V = \{\text{equipo, copa, petaca, tabaco, bigote, duda, bate, vida}\}$. In this set several Spanish words shared occlusive consonants and the Spanish vowels. In every performance we have taken the set of observations from a total set of 11 symbols according to spectral bands techniques. The set of symbols is: $\{p, t, k, b, d, g, a, e, i, o, u\}$.

3. For the training phase ten different speakers were taken to pronounce each

word of the vocabulary three times. This provided a total of $10 \times 8 \times 3$ sound traces. When the system had been trained it was time for evaluation. For that we took samples of another nine different speakers — they were not the same speakers as those of the training phase —. They pronounced each word of the vocabulary three times obtaining a total of $9 \times 8 \times 3$ sound traces.

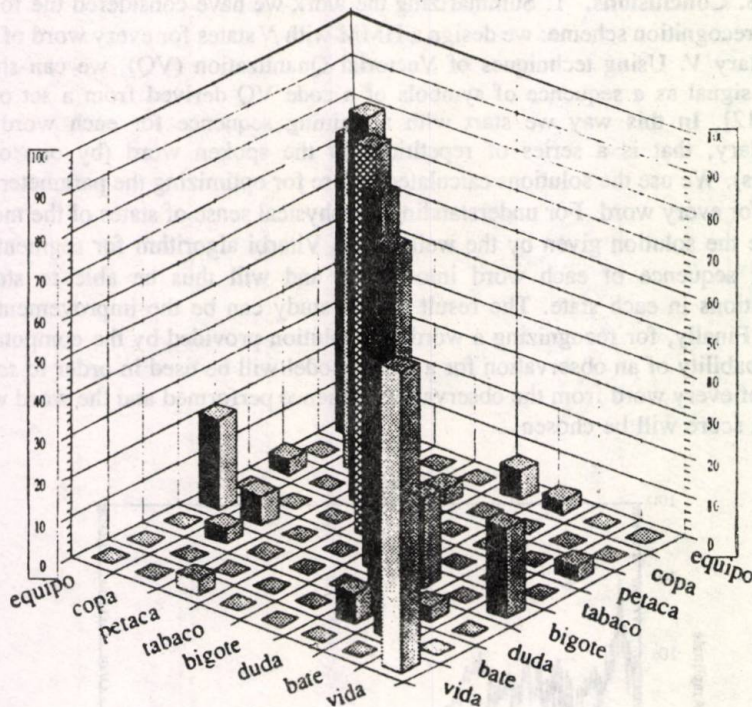


Fig. 7 — Results for recognition of vocabulary formed by eight words, through HMM.

4. Analysis of the results is shown in Figs. 6 and 7. In Fig. 7 we show the percentages of prediction for each of the eight words. Given a word, for instance *equipo*, our recognizing system has guessed the right word in 74.0741 % of the cases, while in 0 % it has given the word *copa*, 3.7037 % the word *petaca*, 0 % the word *tabaco*, 22.2222 % the word *bigote*, 0 % the word *duda*, 0 % the word *bate*, and, finally, 0 % the word *vida*.

The results of Hidden Markov Models we have developed are similar, in the sense of scoring, to those obtained in the different neuronal nets models, except with the multilayer perceptron nets model with which the results are rather better [1], [3].

5. Some basic speech signal processing problems still remain. Sensitivity to

background acoustic noise, changes in the channel characteristics, and misrecognitions of out-of-vocabulary responses all remain difficult problems in speech recognition applications. As our ability to represent a real acoustic signal increases, the need to reject spurious input becomes more acute. While the techniques work well in noise-free environments, demonstration of high performance on even the simplest tasks in operational environments remains a challenge.

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University of Salamanca,
Department of Pure and Applied Mathematics
and
Department of Applied Physics,
Salamanca, Spain

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MODELE MARKOV ASCUNSE ÎN RECUNOAȘTEREA AUTOMATĂ A VORBIRII

O aplicație independentă de vorbitor la un vocabular de cuvinte spaniole

(Rezumat)

Se dezvoltă un sistem hibrid de recunoaștere automată a vorbirii pentru cuvinte spaniole izolate, independent de vorbitor, folosind Modele Markov Ascunse (HMM). În sistemul propus s-a efectuat un stadiu de antrenament pentru modele cu eşantioane sonore provenite de la mai mulți vorbitori (jumătate bărbați și jumătate femei), iar pentru testarea modelelor s-a utilizat un alt grup de vorbitori, diferit de primul.

Rezultatele sunt similare celor obținute cu diverse modele de rețele neurale, dar rămân încă inferioare celor produse de rețelele perceptron.

ABOUT THE NON-LINEAR BEHAVIOUR OF SOME DUFFING TYPE OSCILLATING SYSTEMS

BY

LIVIU BĂDELIȚĂ and VIOREL STANCU

Abstract. A numerical investigation is carried on the nonlinear behaviour of a Duffing type oscillating system. The influence on the system response of the nonlinear restoring force is studied in the phase space, on the time series as well as on the resonance curves. Also investigated is the influence of the equation parameters on the system transit time.

Key words: Duffing equation, nonlinear oscillation, limit cycle, bifurcation, transit time, chaotic oscillation.

1. Introduction. Despite all the efforts — there have already been over 70 years of theoretical and experimental research — and the important number of works concerning this type of oscillators, there has not been reached yet a global description of the Duffing equations' dynamics.

The first significant results, dating from 1918, belong to the Georg Duffing; they pointed out both the frequency-amplitude dependence and the jump phenomenon (or hysteresis). Subsequent research, because the lack of an exact analytical solution, pursued the application of some approximate analytical methods, generally of perturbation type, and the joining of theoretical and experimental results. The computer's development facilitated the extension of the investigated area, the precision growth of the results, along with a decrease of the time computing. New methods of study have been developed and applied to the state space, especially from a topologic point of view.

In spite of that, there are few things known about, for instance, the shape of the resonance curve in case of the external force amplitude's average and high values. The very concept of resonance, developed in connection with the forced oscillators behaviour, is not defined well enough in the general theory of dynamic dissipative systems. The bifurcations, specific to the behaviour of non-linear oscillators, have to be connected with the concept of resonance, for a better outline of the latter.

The numerical investigation that we have made is meant to contribute to a more profound understanding of the behaviour of the equation

$$\begin{aligned} (1) \quad & \ddot{u} + q\dot{u} + \alpha u + \varepsilon u^3 = g \cos \Omega t \\ \text{solution, or of the equivalent system} \\ (2) \quad & \dot{u}_2 = -qu_2 - \alpha u_1 - \varepsilon u_1^3 + g \cos 2\pi u_3, \\ & \dot{u}_3 = \frac{\Omega}{2\pi}. \end{aligned}$$

The values of q , α , ε , g and Ω parameters have been chosen from case to case, according to the pursued objective. However, we can say that, in most of the studied cases, α and ε have been taken positive, situation corresponding to a spring whose restoring force is higher than the linear elastic force or whose potential energy is proportional to the fourth power of the co-ordinate

$$U(x) = \alpha x^2 + \varepsilon x^4.$$

The case is similar to that of a particle in a potential well having the form mentioned above symmetrical, with very high walls.

2. The Influence of the Non-linear Factor. *The first objective of this paper is to notice in what way the nonlinearity of the system chosen for study influences its behaviour in the phase space, modifies the shape of the time series for the co-ordinate u and that of the resonance curves.* To do this we have chosen the following values of the parameters:

$$(3) \quad q=1, \quad \alpha=1, \quad g=4, \quad \Omega=\frac{2}{3},$$

which we have kept constant and varied ε , the non-linear term coefficient. We have given a large number of positive values to the later, some of the intervals having been very accurately investigated. The extreme values have been $\varepsilon=0$ and $\varepsilon=1000$. We have plot the corresponding phase spaces and time series for all these values. We have also plot for the same values the resonance curves (the response's amplitude versus the frequency excitation).

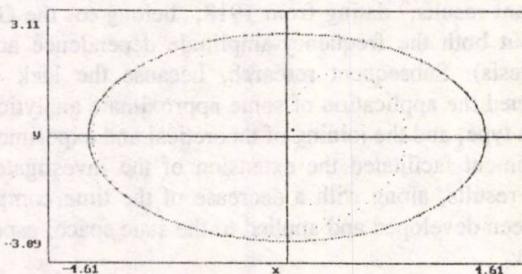


Fig. 1

Out of the plotted diagrams we have selected ten as it follows: three phase space diagrams, Figs. 1 - 3, with the three corresponding time series, Figs. 4 - 6 plotted respectively for the following values:

(4) $\varepsilon=0, \varepsilon=25, \varepsilon=50,$
and four resonance curves, Figs. 7 - 10, plotted respectively for

(5) $\varepsilon=0, \varepsilon=-50, \varepsilon=50, \varepsilon=100.$

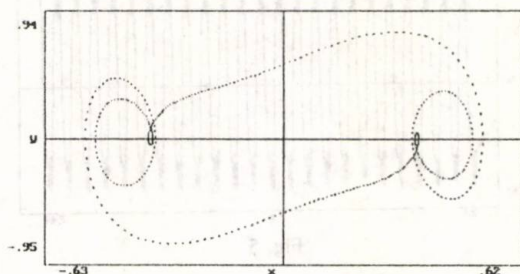


Fig. 2

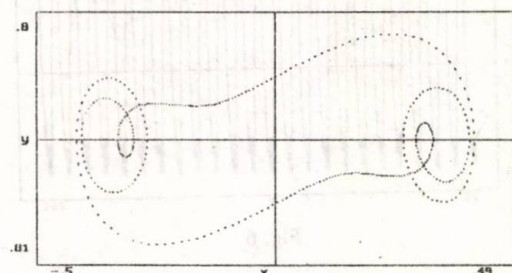


Fig. 3

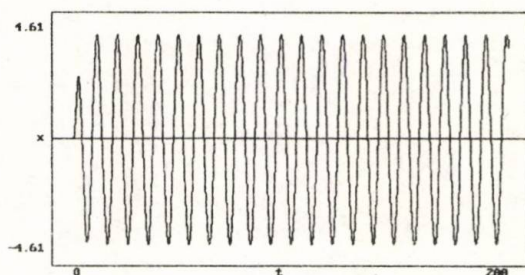


Fig. 4

Analysing these results, we have reached the following conclusions:

- 1) *the trajectory of the representative point from the phase space tends, after a longer or a shorter time interval named transient, to a close orbit, to a limit cycle;*
- 2) *the limit cycle decreases as the non-linear term coefficient's value increases from about ± 3.1 on y and ± 4.6 on x when $\varepsilon=0$ at ± 0.8 on y and ± 0.5 on x when*

$\varepsilon=50$ units. The potential well grows deeper and its walls get closer and closer;

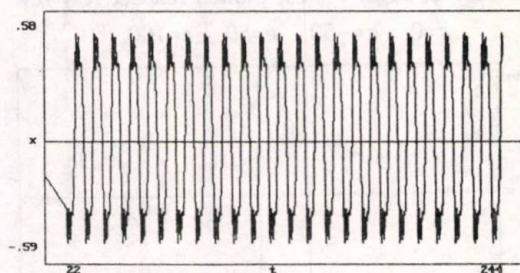


Fig. 5

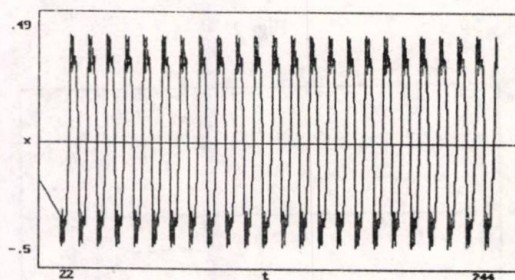


Fig. 6

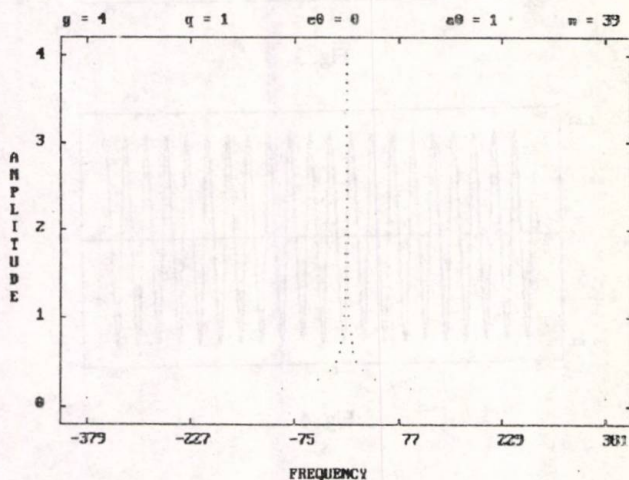


Fig. 7

3) the time series because of the constant amplitude of oscillations, confirms that the oscillator reached the limit cycle;

4) the amplitude of the solution $u=u(t)$, decreases from 4.6 to 0.5 when ε

increases from 0 to 50 units, according to conclusion 2), concerning the potential energy of the system;

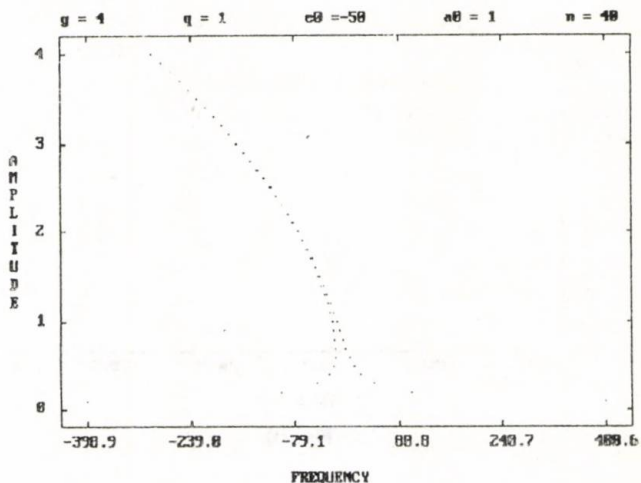


Fig. 8

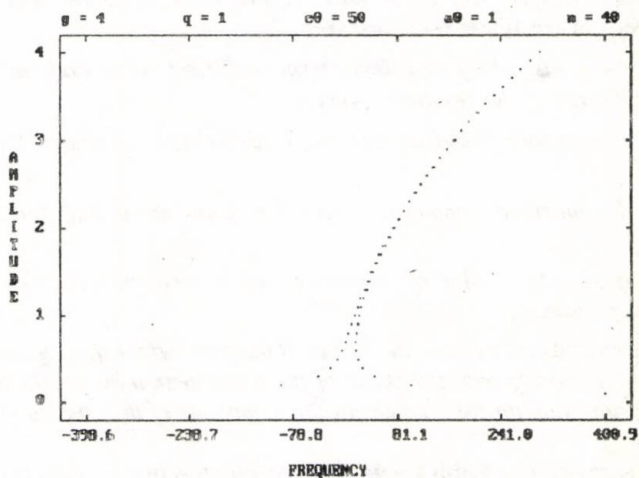


Fig. 9

5) the period of the motion is constant and equal, in the first approximation, with that of the external force ($T=9.25$);

6) the split of time series tops is perfectly correlated with the number of spirals made by the representative point along the limit cycle, as one can infer from the comparative study of phase diagrams and time series;

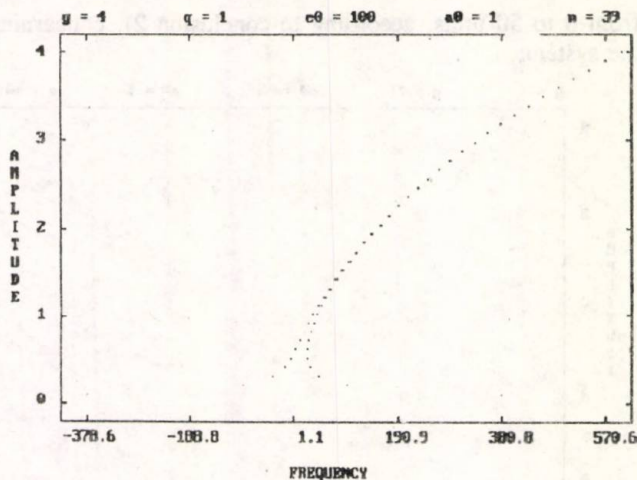


Fig. 10

7) for negative values of ε the resonance curves slope to left and for positive values to right (Figs. 8-10) symmetrically. For $\varepsilon=0$ the curve corresponds to the classic pattern of the linear oscillator (Fig. 7);

8) the increase of the non-linear term coefficient value leads to the increase of the sloping degree of the resonance curve;

9) the resonance frequency does not modify unless the external force amplitude modifies;

10) the maximum amplitude does not grow more than the external force amplitude;

11) to a certain value of frequency may correspond one, two or even three values of amplitude.

Consequently, if we keep the values of the parameters q , α , g and Ω as we have chosen and we modify only the value of the non-linear term coefficient, we cannot notice important qualitative changes in the behaviour of the studied oscillator.

3. Reaching The Limit Cycle. The second objective pursued is concerned with the transit time of the system, respectively with the necessary time for the oscillator to reach the limit cycle.

Modifying the non-linear term coefficient values was not enough for reaching certain conclusions and, consequently, we have operated on all the five parameters.

We have to notice that they qualitatively influence differently the system's behaviour when their change take place in the same way. Generally we can state:

i) if ε , the nonlinearity coefficient, increases, the oscillatory motion limits itself. The same happens if the damping factor increases;

ii) if the external force amplitude's values and its frequency increase, the motion amplifies. The statements rely are based on the study of the phase spaces diagrams and the corresponding time series.

The final motion is, however, the result of the action of the whole set of parameters, of the relations established among their values, their modification affecting, sometime decisively, its character.

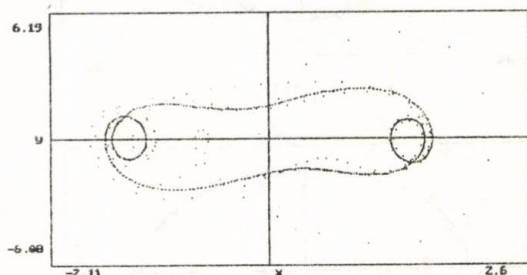


Fig. 11

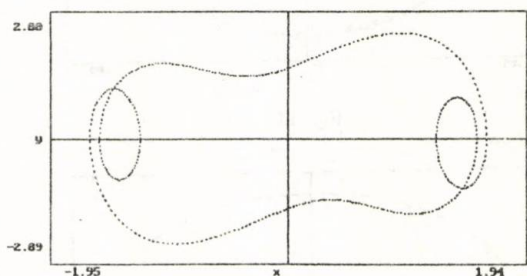


Fig. 12

Starting from these assessments we have chosen convenient values for the system parameters and studied the way in which the transit time is influenced by their values. For Figs. 11 and 12 the parameters' value are

$$(6) \quad q=0.55, \quad \alpha=8, \quad \varepsilon=2, \quad g=24, \quad \omega=1.$$

In Fig. 11 we have left aside the first 200 points and represented the next 2000. One can notice that there is another series of points within the limit cycle. So the transit time is superior to the assumed value. Subsequent attempts have shown that if we leave aside the first 500 points we get a clean limit cycle. The diagram in Fig. 12 is plotted under these circumstances.

The parameters modification according to the next set of values leads to the increase of the transit time. For

$$(7) \quad q=0.05, \quad \alpha=1, \quad \varepsilon=0.1, \quad g=0.8, \quad \omega=5$$

we have plotted the diagrams in Figs. 13 — 15. The first is built with the help of the

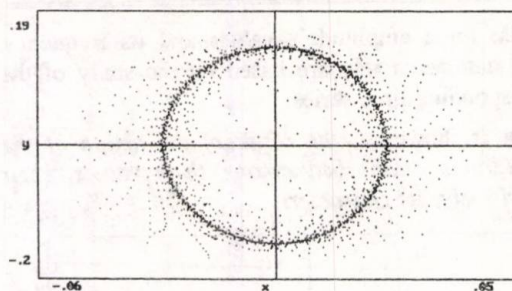


Fig. 13

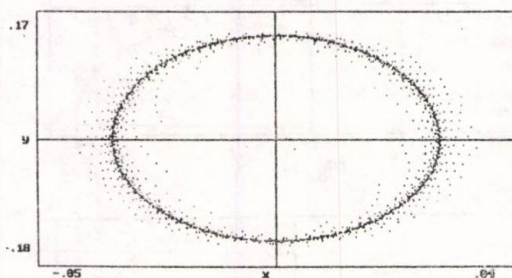


Fig. 14

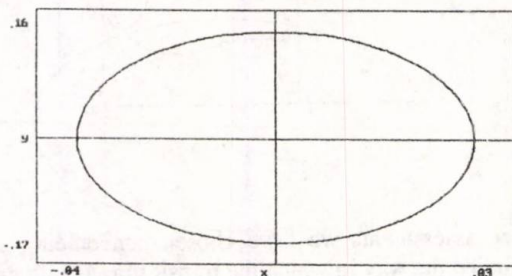


Fig. 15

first (about) 2000 points, the second with the help of 2000 points, too, but with the first 500 points having been left aside. In both situations the limit cycle is visible and the transient states present. The removal of the first (about) 1000 points off the diagram leads to a limit cycle clearly outlined, out of which the transient states have been eliminated.

For the following set of values:

$$(8) \quad q=0.01, \quad \alpha=0.5, \quad \varepsilon=0.05, \quad g=0.1, \quad \omega=10,$$

the transit time increases more so that the first 2222 points neglected are not enough

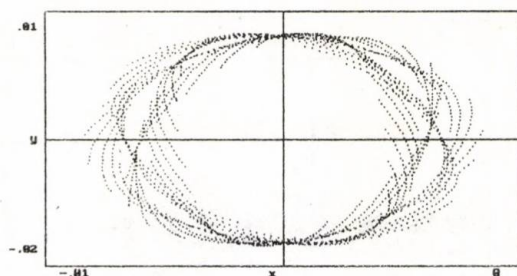


Fig. 16

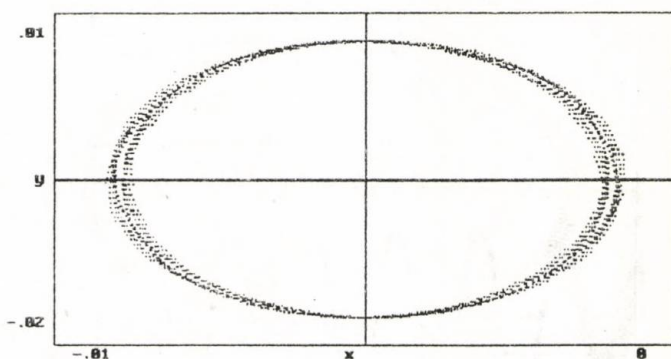


Fig. 17

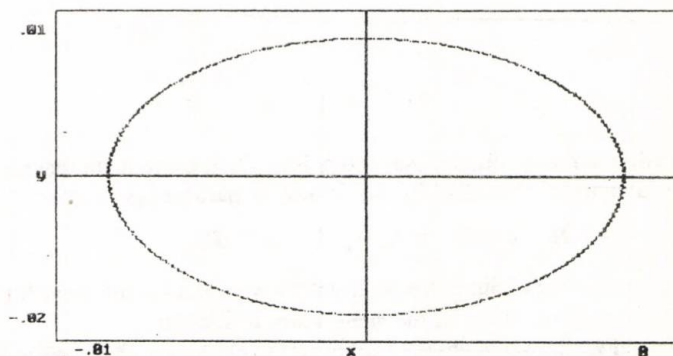


Fig. 18

to outline the limit cycle, nor are the first 5555 (Figs. 16 – 17). The living of the first about 7700 and the plotting of the next 2200 lead us to the pursued result. Fig. 18 is

got under these circumstances.

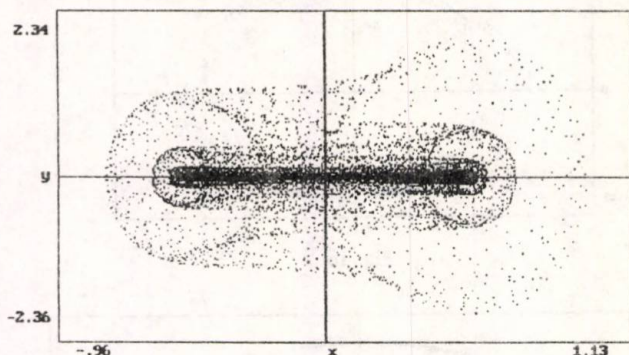


Fig. 19

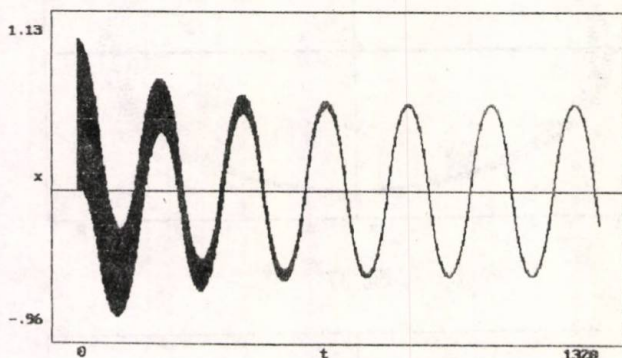


Fig. 20

Fig. 19 together with the time series from Fig. 20 represent the case of an oscillator whose behaviour is controlled by the following parameters' values:

$$(9) \quad q=0.01, \quad \alpha=10, \quad \varepsilon=6, \quad g=8, \quad \omega=0.03.$$

The phase space suggests a three-dimensional structure with axial symmetry, the end of which tapers and repeatedly turns, at the same time, inside out.

The limit cycle does not seem obvious although the phase has diagram has been plotted with the help of 3000 points, the first 10000 being neglected. The time series shows how the superposition of an oscillation with frequency $\omega=0.03$, much lower than the oscillator's eigenfrequency ($\alpha=10$) leads, however, to a periodical motion, to a limit cycle, but after a very long transient phase, actually infinite.

Accordingly, the transit time of the Duffing oscillator defined by Eq. (1) increases whenever the parameters values (q and ε) decrease and g and ω increase.

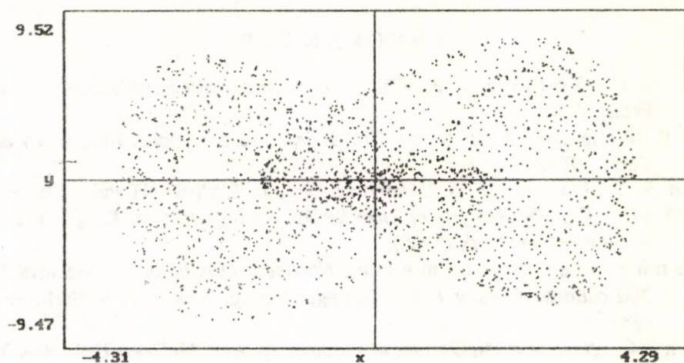


Fig. 21

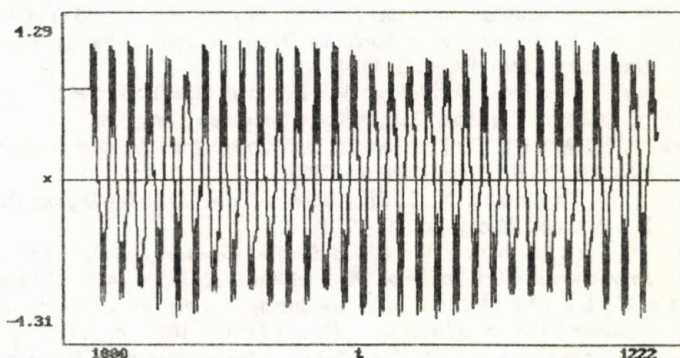


Fig. 22

Given some exceptions, like that illustrated by Figs. 21 and 22, when the oscillator's behaviour is completely disorderly, chaotic, its motion will tend toward a limit cycle, even its external force frequency is very low compared with the oscillator's eigenfrequency.

Remark. For plotting the diagrams we have used the software package CHAOS-AN INTERACTIVE PROGRAM-PACKAGE FOR THE ANALYSIS OF CHAOTIC AND TURBULENT DYNAMICAL SYSTEMS, developed by the authors themselves, implemented on a PC 486 computer, IBM compatible. The Duffing equation has been numerically integrated with the help of a Runge-Kutta method of the fourth order with self-adjustable step.

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Technical University "Gh. Asachi", Jassy,
Department of Physics

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ASUPRA COMPORTĂRII NELINIARE A UNUI OSCILATOR DE TIP DUFFING

(Rezumat)

S-a efectuat o investigație numerică a comportării neliniare a unui oscilator de tip Duffing. S-a studiat influența forței de revenire neliniară asupra comportării sistemului în spațiul fazelor, asupra seriilor de timp și asupra curbelor de rezonanță. De asemenea s-a cercetat influența parametrilor ecuației asupra timpului de tranziție al sistemului.

ON EKMAN PROBLEM FOR LINEAR VISCOELASTIC FLUIDS

BY

A. K. GHOSH and A. R. KHAN

Abstract. The paper intends to describe solutions corresponding to the Ekman problem for viscoelastic fluids of Oldroyd type. The inquiries are made about the exact solutions for the steady and the unsteady velocity fields. In the steady situation the Ekman layer in a viscoelastic fluid resembles the same structure as that of the classical Ekman layer. The effects of the fluid elasticity and the rotation on the unsteady motion are examined through small and large-time solutions. It results that the fluid elasticity modifies the flow-structure only at small values of time while rotation influences the flow at large values of time.

Key words: linear viscoelastic fluid, Ekman problem, Ekman layer, velocity field, transient motion.

1. Introduction. One of the most fundamental problems on the spin-up motion in an incompressible, rotating viscous fluid is the Ekman problem, the solution of which is already incorporated in various text books including those of Green span [1] and Batchelor [2]. This problem concerns with the development of unsteady motion in a semi infinite expanse of incompressible viscous fluid bounded by an infinite, rigid, impermeable disk when both the fluid and the disk are in a state of rigid body rotation with a constant angular velocity about an axis normal to the disk and at some initial instant the rotation rate of the disk is slightly increased.

The object of study was to examine various aspects of the unsteady motion before the new state of solid-body rotation is arrived. From the initial value investigation of the problem it has been shown that the transient motion consists of the development of Ekman layer, spin-up, the decay of inertial oscillations by viscous diffusion and the generation of a purely vertical suction towards the boundary layer, called, the Ekman suction, throughout the main body of the fluid. As a sequel to this, Benton and Lopper [3] discussed the corresponding hydromagnetic problem and Deb Nath and Basu [4] examined a similar problem with Non-Newtonian fluids of Walters type [5].

In the following, Ekman problem for a linear viscoelastic fluid of Oldroyd type [6] has been constructed and solved. The present problem is concerned with the analysis of the unsteady motion generated in an incompressible linear viscoelastic fluid of Oldroyd type bounded by an impervious rigid disk when both the fluid and the disk are in a state of solid-body rotation with a constant angular velocity about an axis normal to the disk and at some initial instant the unsteady motion is created in the revolving viscoelastic system by slightly increasing the angular speed of rotation of the disk. The analysis is carried out to examine the various physical processes involved in the ensuing transient motion approaching the steady state condition. The problem is solved as an initial and boundary value problem. The exact expressions for the unsteady and the steady velocity fields are obtained. The small and large time solutions are studied with a view to discuss the effects of rotation and the elasticity on the flow. The structure of the boundary layer and the Ekman suction velocity are also found with physical significance.

2. Mathematical Formulation. It was shown by Oldroyd [6] that under the conditions of slow motion, the properties of many viscoelastic fluids can be specified by three constants, η_0 with the dimension of viscosity and λ_1, λ_2 with the dimensions of time. The equation of state in such a situation takes the form

$$(2.1) \quad p_{ik} = p'_{ik} - p - \delta_{ik},$$

$$(2.2) \quad \left[1 + \lambda_1 \frac{\partial}{\partial t} \right] p'_{ik} = 2\eta_0 \left[1 + \lambda_2 \frac{\partial}{\partial t} \right] e_{ik},$$

$$(2.3) \quad e_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}),$$

where u_i denotes velocity vector; δ_{ik} is the Kronecker delta, p'_{ik} is the part of the stress tensor related to the change of shape of the material and p is the isotropic fluid pressure. The above model describes the liquid ($e_{ii}=0$) exhibiting Weissenberg effect if $\eta_0 > 0$ and $\lambda_1 > \lambda_2$ and viscous liquid if $\eta_0 > 0$ and $\lambda_1 = \lambda_2$.

We consider the situation in which an incompressible viscoelastic fluid is confined to the upper half plane $z > 0$ by an impervious rigid disk at $z=0$. Initially, both the fluid and the disk rotate rigidly with one another with a constant angular velocity Ω about the z -axis which is normal to the disk. At $t > 0$, the unsteady motion is generated in the revolving system by slightly increasing the angular velocity of the disk so that it creates a small deviation from the initial state of solid-body rotation. The magnitude of velocity of the unsteady motion is then small compared to basic rotation speed. Consequently, the non-linear terms in the equations of motion are negligible and the linearized theory for rotating fluids is applicable in the case of present problem.

In view of the Eqs. (2.1) and (2.2) and the physical situation of the fluid motion stated above, we find that the unsteady motion of a viscoelastic fluid with respect to the rotating frame of reference is governed by the following equations with usual notations:

$$(2.4) \quad \left[1 + \lambda_1 \frac{\partial}{\partial t} \right] \left[\frac{\partial \mathbf{q}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{q} \right] = -\frac{1}{\rho} \left[1 + \lambda_1 \frac{\partial}{\partial t} \right] \nabla P + \nu \left[1 + \lambda_2 \frac{\partial}{\partial t} \right] \nabla^2 \mathbf{q},$$

and

$$(2.5) \quad \nabla \cdot \mathbf{q} = 0,$$

where $\mathbf{q} = (u, v, w)$ represents the fluid velocity, P is the fluid pressure including the centrifugal term; $P, \nu, \lambda_1, \lambda_2$ are all assumed constant throughout the flow field and $\boldsymbol{\Omega} = (0, 0, \Omega)$ is the angular velocity of the coordinate system. The initial and boundary conditions appropriate for the present problem are given by

$$(2.6) \quad (u, v, w) = (0, 0, 0) \quad \text{at } t \leq 0, z > 0;$$

$$(2.7) \quad \begin{cases} v = \varepsilon_1 r \Omega \\ u, w = 0, 0 \end{cases} \quad \text{at } z = 0, t > 0;$$

$$(2.8) \quad (u, v) \rightarrow (0, 0) \quad \text{as } z \rightarrow \infty.$$

Introducing cylindrical polar coordinates we write

$$(2.9) \quad \mathbf{q} = r V(z, t) \boldsymbol{\theta} - \nabla \times \{ \boldsymbol{\theta} \psi(r, z, t) \}, \quad \psi(r, z, t) = r \chi(z, t),$$

where $\boldsymbol{\theta}$ is the unit vector in the θ -direction and $V(z, t)$ is the angular velocity of the instantaneous motion and $\psi(r, z, t)$ is the stream function.

Applying (2.9) in (2.4) and (2.6) - (2.8), we get

$$(2.10) \quad \left[1 + \lambda_1 \frac{\partial}{\partial t} \right] \left[\frac{\partial Q}{\partial t} - 2i\Omega Q \right] = \nu \left[1 + \lambda_2 \frac{\partial}{\partial t} \right] \frac{\partial^2 Q}{\partial z^2},$$

$$(2.11) \quad \left[\frac{\partial \chi}{\partial z}, V, \chi \right] = (0, 0, 0) \quad \text{at } t \leq 0, z < 0,$$

$$(2.12) \quad \begin{cases} \frac{\partial \chi}{\partial z}, \chi = (0, 0) \\ V = \varepsilon_1 \Omega \end{cases} \quad \text{at } z = 0, t > 0,$$

$$(2.13) \quad \left[\frac{\partial \chi}{\partial z}, V \right] = (0, 0) \quad \text{as } z \rightarrow \infty, t > 0,$$

and

$$Q = \frac{\partial V}{\partial z} + i \frac{\partial^2 \chi}{\partial z^2}.$$

3. Solution of the Problem. We take the dimensionless quantities as

$$(3.1) \quad z' = \frac{z}{L}, \quad t' = \Omega t, \quad V' = \frac{V}{\varepsilon_1 \Omega}, \quad \chi' = \frac{\chi}{\varepsilon_1 \Omega L},$$

$$Q' = \frac{QL}{\varepsilon_1 \Omega}, \quad \lambda_1' = \lambda_1 \Omega, \quad \varepsilon = \frac{\lambda_2}{\lambda_1}, \quad (\lambda_2 < \lambda_1), \quad E = \frac{\nu}{\Omega L^2},$$

where $L\Omega^{-1}$ are the typical length and time for a particular motion.

Introducing (3.1) in (2.10) and (2.11) – (2.13), after dropping the primes, we get

$$(3.2) \quad \left[1 + \lambda_1 \frac{\partial}{\partial t}\right] \left[\frac{\partial Q}{\partial t} - 2iQ \right] = E \left[1 + \lambda_1 \varepsilon \frac{\partial}{\partial t}\right] \frac{\partial^2 Q}{\partial z^2}$$

with

$$(3.3) \quad \left[\frac{\partial \chi}{\partial z}, \chi, V \right] = (0, 0, 0) \quad \text{at } t \leq 0, z > 0,$$

$$(3.4) \quad \begin{cases} \left[\frac{\partial \chi}{\partial z}, \chi \right] = (0, 0) \\ V(z, t) = 1 \end{cases} \quad \text{at } z = 0, t > 0,$$

$$(3.5) \quad \left[\frac{\partial \chi}{\partial z}, V \right] = (0, 0) \quad \text{as } z \rightarrow \infty, t > 0.$$

Using the usual Laplace transform technique, the transformed solutions corresponding to the set equations (3.2) – (3.5) are obtained as

$$(3.6) \quad \bar{V} = \frac{1}{2s} (e^{-\mu_1 z} + e^{-\mu_2 z}),$$

$$(3.7) \quad \frac{\partial \bar{\chi}}{\partial z} = \frac{i}{2s} (e^{-\mu_1 z} - e^{-\mu_2 z}),$$

and

$$(3.8) \quad \bar{\chi} = \frac{i}{2s} \left[\frac{1}{\mu_1} - \frac{1}{\mu_2} \right] - \left[\frac{e^{-\mu_1 z}}{\mu_1} - \frac{e^{-\mu_2 z}}{\mu_2} \right],$$

where

$$(3.9) \quad \mu_1, \mu_2 = \left[\frac{(1 + \lambda_1 s)(s \pm 2i)}{E(1 + \lambda_1 \varepsilon s)} \right]^{1/2}$$

and s is the Laplace transform variable.

The inverse transforms, after performing contour integrations, provide the component of fluid velocity as

$$(3.10) \quad \begin{aligned} \frac{v}{r} = V(z, t) = & e^{-\xi} \cos \xi - \frac{1}{2\pi} \left\{ \int_{1/\lambda_1}^{1/\lambda_1 \varepsilon} \frac{e^{-xt}}{x} \left[\sin \xi \left(\frac{(1 - x\lambda_1)(x - 2i)}{1 - x\lambda_1 \varepsilon} \right)^{1/2} + \right. \right. \\ & \left. \left. + \sin \xi \left(\frac{(1 - x\lambda_1)(x + 2i)}{1 - x\lambda_1 \varepsilon} \right)^{1/2} \right] dx + \int_0^\infty \frac{e^{-xt - 2it}}{x + 2i} \sin \xi \left(\frac{[1 - \lambda_1(x + 2i)]x}{1 - \lambda_1 \varepsilon(x + 2i)} \right)^{1/2} dx + \right. \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty \frac{e^{-xt+2it}}{x-2i} \sin \xi \left[\frac{[1-\lambda_1(x-2i)]x}{1-\lambda_1 \varepsilon(x-2i)} \right]^{1/2} dx \Bigg\}, \\
 & \frac{u}{r} = \frac{\partial \chi}{\partial z} = e^{-\xi} \sin \xi - \frac{i}{2\pi} \left\{ \frac{1}{\lambda_1} \frac{e^{-xt}}{x} \left[\sin \xi \left[\frac{(1-x\lambda_1)(x-2i)}{1-\lambda_1 \varepsilon x} \right]^{1/2} - \right. \right. \\
 & \left. \left. - \sin \xi \left[\frac{(1-x\lambda_1)(x+2i)}{1-\lambda_1 \varepsilon x} \right]^{1/2} \right] dx + \int_0^\infty \frac{e^{-xt-2it}}{x+2i} \sin \xi \left[\frac{[1-\lambda_1(x+2i)]x}{1-\lambda_1 \varepsilon(x+2i)} \right]^{1/2} dx - \right. \\
 & \left. - \int_0^\infty \frac{e^{-xt+2it}}{x-2i} \sin \xi \left[\frac{[1-\lambda_1(x-2i)]x}{1-\lambda_1 \varepsilon(x-2i)} \right]^{1/2} dx \right\},
 \end{aligned}
 \tag{3.11}$$

$$\begin{aligned}
 w = -2\chi(z, t) = E^{1/2} & \left\{ -1 + e^{-\xi} (\sin \xi + \cos \xi) + \right. \\
 & + \frac{i}{\pi} \int_0^\infty \frac{e^{-xt+2it}}{x+2i} \left[\frac{1-\lambda_1 \varepsilon(x+2i)}{[1-\lambda_1(x+2i)]x} \right]^{1/2} \left[1 - \cos \xi \left[\frac{[1-\lambda_1(x+2i)]x}{1-\lambda_1 \varepsilon(x+2i)} \right]^{1/2} \right] dx - \\
 & \left. - \frac{i}{\pi} \int_0^\infty \frac{e^{-xt-2it}}{x-2i} \left[\frac{1-\lambda_1 \varepsilon(x-2i)}{[1-\lambda_1(x-2i)]x} \right]^{1/2} \left[1 - \cos \xi \left[\frac{[1-\lambda_1(x-2i)]x}{1-\lambda_1 \varepsilon(x-2i)} \right]^{1/2} \right] dx \right\},
 \end{aligned}
 \tag{3.12}$$

where $\xi = z/\sqrt{E}$.

The results (3.10) - (3.12) describe the general features of the velocity field in a linear viscoelastic fluid during its spin-up motion. Clearly, the effect of the elasticity of the fluid remains confined with the transient part of the solution. Since the transient parts are decaying with time, the effect of the elasticity on the flow gradually diminishes with time and the ultimate flow becomes independent of the effect of the elasticity of the fluid. Consequently, the structure of the Ekman layer both in viscous and viscoelastic fluids becomes identical when the steady condition is reached.

The steady-state results, in the limit $t \rightarrow \infty$, are given by

$$(3.13)_{a,b} \quad \frac{v}{r} = e^{-\xi} \cos \xi, \quad \frac{u}{r} = e^{-\xi} \sin \xi,$$

$$(3.14) \quad w = \sqrt{E} [-1 + e^{-\xi} (\cos \xi + \sin \xi)],$$

which are exactly identical to those given by Greenspan [1] in course of his

analysis on viscous fluid. In the limit $\xi \rightarrow \infty$ we find from (3.14) that there exists a purely vertical inflow due to the boundary layer suction which is often called *Ekman suction velocity* and is given by $w = -\sqrt{E}$ for both viscous and viscoelastic fluids. Finally, the thickness of the Ekman layer which is attained in the steady state is of order $\sqrt{\nu/\Omega}$, and is obviously the same for both viscous and viscoelastic fluids.

In particular, when $\varepsilon = 1$, the exact solution for the velocity components corresponding to the classical Ekman problem are recovered from (3.10) – (3.12). The results are

$$(3.15) \quad \frac{v}{r} = \Re e \left[e^{-\xi\sqrt{2i}} + F_2(\xi, t) \right],$$

$$(3.16) \quad \frac{u}{r} = -\Im m \left[e^{-\xi\sqrt{2i}} + F_2(\xi, t) \right],$$

$$(3.17) \quad w = 2E^{1/2} \Im m \left[\frac{\operatorname{erf}\sqrt{2it}}{\sqrt{2i}} - \frac{e^{-\xi\sqrt{2i}}}{\sqrt{2i}} + \frac{F_1(\xi, t)}{\sqrt{2i}} \right],$$

where $\xi = z/\sqrt{E}$ and

$$(3.18) \quad F_{1,2}(\xi, t) = \frac{1}{2} \left[e^{\xi\sqrt{2i}} \operatorname{erfc} \left[\sqrt{2it} + \frac{\xi}{2\sqrt{t}} \right] \pm e^{-\xi\sqrt{2i}} \operatorname{erfc} \left[\sqrt{2it} - \frac{\xi}{2\sqrt{t}} \right] \right].$$

The expressions (3.15) – (3.17) coincide exactly with the findings of Greenspan [1].

4. Early-Time Solutions. The early-time behaviour of the fluid motion is investigated with assumption that $\lambda_1 s$ and $\lambda_1 \varepsilon s$ are large when s is large.

The transformed results (3.6) – (3.8) are then approximated for large values of s to obtain the small-time solutions by inversions. The solutions for the components of the fluid velocity then become:

$$(4.1) \quad \frac{v}{r} \approx (1 - 4\eta^2 t) \operatorname{erfc} \eta + \frac{8m\eta^2}{\sqrt{\pi}} e^{-\eta^2 t^{3/2}},$$

$$(4.2) \quad \frac{u}{r} \approx -4\eta \left[\eta \operatorname{erfc} \eta - \frac{1}{\sqrt{\pi}} e^{-\eta^2} \right] t,$$

$$(4.3) \quad w \approx 2\sqrt{E\varepsilon} \Im m \left\{ \frac{e^{mt-it}}{i\sqrt{i-m}} \operatorname{erf}(i\sqrt{(i-m)t}) - \frac{i}{2} \frac{e^{mt-it}}{\sqrt{i-m}} \left[e^{2i\eta\sqrt{(i-m)t}} \operatorname{erfc}(\eta + i\sqrt{(i-m)t}) - e^{-2i\eta\sqrt{(i-m)t}} \operatorname{erfc}(\eta - i\sqrt{(i-m)t}) \right] \right\},$$

where

$$\eta = \frac{z}{2\sqrt{E\epsilon t}} = \frac{\xi}{2\sqrt{\epsilon t}}, \quad m = \frac{1}{2\lambda_1} \left[\frac{1}{\epsilon} - 1 \right], \quad \epsilon = \frac{\lambda_2}{\lambda_1} < 1.$$

These results are valid for non-zero finite values of λ_1 and ϵ and express the fact that the initial stage of the motion consists of a viscoelastic Rayleigh layer of thickness $\sqrt{\nu \epsilon t}$ accompanied by vertical oscillations of frequency Ω whose amplitude and phase are modified by the elasticity of the fluid. The thickness of the initially formed Rayleigh layer is less than that of the Classical Rayleigh layer due to the presence of the elastic parameter. Moreover, the absence of rotation in the higher order terms of (4.1) and (4.2) clearly indicate that unlike elasticity the rotation takes comparatively larger time to influence the motion within the boundary layer. In fact, the early-time behaviour of the viscoelastic fluid is similar to that of viscous fluid whose kinematic viscosity ν is reduced by the factor ϵ .

5. Large-Time Solutions. For many practical viscoelastic fluids λ_1 and $\lambda_1 \epsilon$ are both small. We suppose that s is also small and expand μ_1 and μ_2 for small values of $\lambda_1 s$ and $\lambda_1 \epsilon s$. The approximate values of μ_1, μ_2 are taken as

$$\mu_{1,2} \approx \left[\frac{s \pm 2i}{E} \right]^{1/2} (1 + as + bs^2),$$

where

$$a = \frac{\lambda_1(1-\epsilon)}{2}, \quad b = \frac{\lambda_1^2}{8}(3\epsilon^2 - 2\epsilon - 1).$$

Using these results in (3.6) - (3.8) and inverting them we get

$$\begin{aligned} \frac{v}{r} \approx \Re e \left\{ e^{-\xi\sqrt{2i}} + F_2(\xi, t) + \left[\frac{(a-2ib)\xi}{2\sqrt{\pi}t^{3/2}} - \frac{1}{4\sqrt{\pi}t^{5/2}} [\xi(a\xi^2+3b) - \right. \right. \\ \left. \left. - i\xi^3(2b+3a^2)] \right] e^{-2it-\xi^2/4t} + O\left[\frac{1}{t^{7/2}}\right] \right\}, \end{aligned} \quad (5.1)$$

$$\begin{aligned} \frac{u}{r} \approx \Im m \left\{ e^{-\xi\sqrt{2i}} + F_2(\xi, t) + \left[\frac{(a-2ib)\xi}{2\sqrt{\pi}t^{3/2}} - \frac{1}{4\sqrt{\pi}t^{5/2}} [\xi(a\xi^2+3b) - \right. \right. \\ \left. \left. - i\xi^3(2b+3a^2)] \right] e^{-2it-\xi^2/4t} + O\left[\frac{1}{t^{7/2}}\right] \right\}, \end{aligned} \quad (5.2)$$

$$(5.3) \quad w \approx 2E \Im m \left\{ \frac{\operatorname{erf} \sqrt{2i} t}{\sqrt{2i}} - \frac{a+2ia^2-2ib}{\sqrt{\pi t}} e^{-2it} - \frac{a^2-b}{2\sqrt{\pi t}^{3/2}} e^{-2it} - \left[\frac{e^{-\xi \sqrt{2i}}}{\sqrt{2i}} - \frac{F_1(\xi, t)}{\sqrt{2i}} - \frac{a^2-2ia^2-2ib}{\sqrt{\pi t}} e^{-2it-\xi^2/4t} + \frac{a^2+b^2-a+2ia^2-2ib}{2\sqrt{\pi t}^{3/2}} e^{-2it-\xi^2/4t} + O\left(\frac{1}{t^{5/2}}\right) \right] \right\}.$$

The results (5.1) – (5.3) represent the components of the fluid velocity for large values of time when λ_1 and $\lambda_1 \varepsilon$ are non-zero and finite. These results can also be considered for small values of the elastic parameters λ_1 and $\lambda_1 \varepsilon$ when t is finite. Additionally, we find that unlike small time solutions, the fluid oscillates with frequency 2Ω for large values of time and the effect of rotation persists even when $t \rightarrow \infty$.

To investigate the transient approach towards the ultimate steady motion we consider the asymptotic expansion of the complementary error function with complex argument z in the form

$$(5.4) \quad \operatorname{erfc} z \approx \frac{e^{-z^2}}{z\sqrt{\pi}} \quad \text{as } z \rightarrow \infty, \quad \operatorname{erfc}(-z) = 2 - \operatorname{erfc} z.$$

Using (5.4) in (5.1) – (5.3), asymptotic s for larger t has representations

$$(5.5) \quad \frac{v}{r} = \Re e \left[e^{-\xi \sqrt{2i}} - \frac{\xi}{\sqrt{\pi t}} \frac{e^{-2it-\xi^2/4t}}{2it-\xi^2/4t} \right],$$

$$(5.6) \quad \frac{u}{r} = -\Im m \left[e^{-\xi \sqrt{2i}} - \frac{\xi}{\sqrt{\pi t}} \frac{e^{-2it-\xi^2/4t}}{2it-\xi^2/4t} \right],$$

$$(5.7) \quad w = 2E \Im m \left[\frac{\operatorname{erf} \sqrt{2i} t}{\sqrt{2i}} - \frac{e^{-\xi \sqrt{2i}}}{\sqrt{2i}} - \frac{a+2ia^2-2ib}{\sqrt{\pi t}} e^{-2it} + \frac{a-2a^2-2ib}{\sqrt{\pi t}} e^{-2it-\xi^2/4t} + \frac{\sqrt{t}}{\sqrt{\pi} (2it-\xi^2/4t)} e^{-2it-\xi^2/4t} \right].$$

These results clearly reveal that the ultimate boundary layer is established through inertial oscillations of frequency 2Ω which decay ultimately with time. The total transient time required for the establishment of the ultimate steady Ekman layer is $t=2/\Omega$ which follows from the argument of the error function involved in (5.7).

The present analysis thus concludes that as soon as the impulsive change is made on the angular velocity of the disk, its motion is communicated to the fluid through a viscoelastic Rayleigh layer of thickness $\sqrt{\nu \varepsilon t}$ which then starts thickening due to viscous diffusion and ultimately the Ekman layer similar that of the classical Ekman layer of thickness $\sqrt{\nu/\Omega}$ is formed in the viscoelastic fluid through a series of inertial oscillations of frequency 2Ω in time $t=2/\Omega$.

The elasticity of the fluid has a dominant effect only at the initial stage of the motion and this effect gradually diminishes as the transient motion approaches the steady state. As a result, the Ekman layer in viscoelastic fluid resembles the same structure as that of the classical viscous fluid.

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Jadavpur University,
Department of Mathematics,
Calcutta, India

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ASUPRA PROBLEMEI LUI EKMAN PENTRU FLUIDE VÂSCOELASTICE LINIARE

(Rezumat)

Se studiază problema lui Ekman pentru fluide vâscoelastice de tip Oldroyd obținându-se soluții exacte pentru câmpurile de viteze staționar și nestaționar. În cazul staționar, stratul Ekman al fluidului vâscoelastic prezintă structură analogă celei din cazul clasic. Se examinează efectele elasticității fluidului și ale rotației asupra mișcării nestaționare cu ajutorul soluțiilor pentru intervale de timp scurte și intervale lungi. Se stabilește că elasticitatea fluidului modifică structura cărgerii numai pe durate scurte, în timp ce rotația influențează curgerea pe durate mari de timp.

THE KINK SOLUTION OF THE GINZBURG-LANDAU EQUATION. TOPOLOGICAL ANALYSIS

BY

MARICEL AGOP, N. REZLESCU, CĂLIN GH. BUZEA and CRISTINA BUZEA

Abstract. Using the lagrangean formalism, the kink solution of the Ginzburg-Landau equation in the presence of spatial gradients and in the absence of fields is obtained. In this formalism, the kink "energy" is inversely proportional to the coherence length and the topological charges are associated to the total spin of the particle described by the Ginzburg-Landau equation (Cooper pair).

Key words: Ginzburg-Landau equation, Cooper pair, kink solution, coherence length, topological charge, particle spin.

1. The Ginzburg-Landau (GL) equation allows the phenomenological study of the superconducting state [1], [2], the dynamics of superconducting phase by changing the value of the Ginzburg-Landau parameter k from the type - I to the type - II regions [3] the dynamical interactions of two colliding vortices [4], the vortex nucleation and flux flow [5], etc. The successive approximations [6] or numerical [7] methods allowed the study of the spatial variation of the superconducting state parameters.

2. In this paper, using the lagrangean formalism [8], the kink solution of the GL equation in the presence of spatial gradients and in the absence of fields is obtained. In this formalism, the kink "energy" is inversely proportional to the coherence length. Also, a brief topological analysis is performed. The associated topological charges are quantum numbers, being identified to the total spin of the particle described by the Ginzburg-Landau equation (Cooper pair).

Formally, the Ginzburg-Landau equation

$$(1) \quad \xi^2 \frac{d^2 G}{dx^2} + G - G^3 = 0$$

results from the Euler-Lagrange equations for the lagrangean

$$(2) \quad L = -\frac{1}{2}\xi^2 \left[\frac{dG}{dx} \right]^2 - V(G)$$

with the "potential"

$$(3) \quad V(G) = \frac{1}{4}G^2(G^2 - 2).$$

The fundamental state is given by the condition

$$(4) \quad \frac{dV}{dG} = 0,$$

from where

$$(5) \quad G_0 = 0, \quad G_{1,2} = \pm 1.$$

Substituting these values in the second derivative

$$(6) \quad \frac{d^2V(0)}{dG^2} = -1, \quad \frac{d^2V(\pm 1)}{dG^2} = 2,$$

so the $G_{1,2} = \pm 1$ solutions can be associated to the minimum of the energy (Fig. 1). For $G \rightarrow iG$ the function $V(G)$ has a single minimum (Fig. 2).

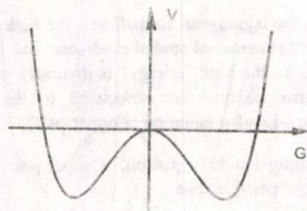


Fig. 1 — The $V(G)$ dependence with a double minima.

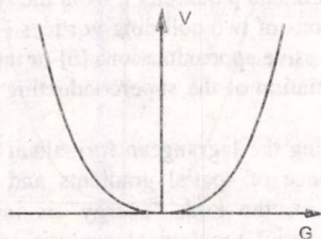


Fig. 2 — The $V(G)$ dependence with a single minimum.

The energy has the form

$$(7) \quad E(G) = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\xi^2 \left[\frac{dG}{dx} \right]^2 + V(G) \right]$$

and the value of the energy relative to the fundamental state becomes

$$(8) \quad E(G) - E(G_{1,2}) = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \xi^2 \left(\frac{dG}{dx} \right)^2 + \frac{1}{4} (G^2 - 1)^2 \right].$$

Because all the terms of Eq. (8) are positive and the integration limits are infinite, the condition of the finite energy for $x \rightarrow \pm \infty$ implies

$$(9) \quad \frac{dG}{dx} = 0$$

and

$$(10) \quad \frac{1}{4} (G^2 - 1) = 0,$$

so that for $x \rightarrow \pm \infty$ the function $G(x)$ tends to its fundamental state values

$$(11) \quad G_V = \pm 1,$$

which we call "vacuum values".

Let integrate Eq. (1) over x :

$$(12) \quad \frac{1}{2} \xi^2 \left(\frac{dG}{dx} \right)^2 = \frac{1}{4} G^4 - \frac{1}{2} G^2 + \frac{1}{2} G_0,$$

with G_0 an integration constant. A second integration of (12) leads to

$$(13) \quad \frac{x - x_0}{\xi} = \int_0^G \frac{dG}{\sqrt{\frac{1}{2} G^4 - G^2 + G_0}},$$

where x_0 is an integration constant. To this solution, with an arbitrary G_0 , corresponds an infinite value of the energy. To obtain a finite value of the energy we impose the restriction (12) which implies, taking into account (12),

$$(14) \quad G_0 = \frac{1}{2}.$$

Substituting (14) in (13) the finite energy solution of (1) is called the "kink":

$$(15) \quad G_k = G(x - x_0) = \pm \operatorname{th} \left[\frac{1}{\sqrt{2} \xi} (x - x_0) \right],$$

with the representation from Fig. 3.

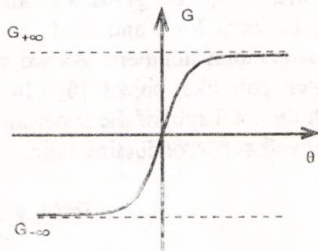


Fig. 3 -- The representation of "kink" solution.

The energy of the kink relative to the fundamental state is

$$(16) \quad E(G_k) - E(G_{1,2}) = \frac{2\sqrt{2}}{3} \frac{1}{\xi}.$$

3. A topological method can be applied because the admissible number of kinks is determined by the topological properties of the symmetry group of the Ginzburg-Landau equation. In this context, the following problems must be solved:

- i) the number of admissible kink solutions determined by the topological properties of the Ginzburg-Landau equation;
- ii) the topological charge.

The kink solutions can be considered as mappings of a spatial zero-sphere S^0 , taken at infinity onto the vacuum manifold of the G-L model. The homotopy group for this model is (the same for the χ model) [8]

$$(17) \quad \pi_0(Z_0) = Z_2,$$

i.e., the model gives rise to two solutions: a constant solution G_V and a kink solution. The homotopic mappings is given in Fig. 4.

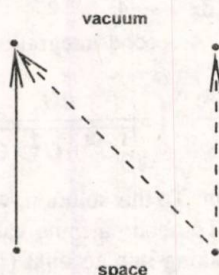


Fig. 4 — Homotopic mapping of the zero-sphere onto the vacuum manifold.

The associated topological charge is [8]

$$(18) \quad S = \frac{1}{2} \int_{-\infty}^{+\infty} dx j_0(x) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dG}{dx} dx = \frac{1}{2} [G(+\infty) - G(-\infty)].$$

The vacuum solution (absence of spatial gradients) and the kink solution can be characterized by attributing to them $S=0$ and $S=1$, respectively. This topological charges may be regarded as quantum numbers. As we know, the Ginzburg-Landau equation describes a Cooper pair-like object [6]. In this case we identify the topological charge $S=0$ with the total spin of the superconducting "particle" and $S=1$ to the total spin of the destroyed superconducting pair.

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Technical University "Gh. Asachi", Jassy,
Department of Physics
and
Institute of Technical Physics, Jassy,
Superconductivity Research Laboratory

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SOLUȚIA "KINK" A ECUAȚIEI GINZBURG-LANDAU.
ANALIZĂ TOPOLOGICĂ

(Rezumat)

Utilizând formalismul lagrangean se obține soluția "kink" a ecuației Ginzburg-Landau în prezența gradientilor spațiali și în absența câmpurilor. În acest formalism, "energia" kink este invers proporțională cu lungimea de coerență, iar sarcinile topologice sunt asociate cu spinul total al particulei descris de ecuația Ginzburg-Landau (pereche Cooper).

APPENDIX

1. The first item is a letter from the Secretary of the State Department to the President, dated January 1, 1900. It contains a report on the progress of the work of the Department during the year 1899. The letter is signed by the Secretary, John Hay.

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THE GRAVITON PHOTOPRODUCTION IN EXTERNAL ELECTROMAGNETIC FIELDS

BY

DORIAN ȚĂTOMIR and ADRIAN SACHELARIE

Abstract. Starting from the principle of minimal coupling in Quantum Gravity and using the S -matrix formalism, the processes of graviton photoproduction in external electromagnetic fields (the Coulomb field of the nucleus, homogeneous electrostatic field and the electric dipole field) are studied and the corresponding differential cross-sections are obtained. The results are in agreement with those obtained by many other authors in a different manner. The process of transformation of photon into graviton in the electric dipole field is not studied in the literature, but in our opinion it exists, the gravitational wave emission appearing in this case only as a quantum effect.

Key words: quantum gravity, graviton, graviton photoproduction, gravitational wave emission.

In the present paper, using the S -matrix formalism [1] and Gupta's linear approximation [2]¹

$$(1) \quad \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - k y^{\mu\nu},$$

the processes of graviton photoproduction in external electromagnetic fields are studied. Here $g^{\mu\nu}$, $\eta^{\mu\nu}$ and $y^{\mu\nu}$ are the metric tensor, the Minkowski tensor-diag. (1, -1, -1, -1) and the tensor of the weak gravitational field, respectively, $g = \det(g_{\mu\nu})$ and

$$(1') \quad k = \sqrt{16\pi G}$$

(in natural units $\hbar=c=1$), G being the Newton constant.

Following M i t s k e v i c h's ideas [3] and using the principle of minimal coupling [4], the expression of Lagrangian for the Maxwell field in curved space becomes

¹) Components of 3-vectors are labelled by Roman indices, while components of 4-vectors carry Greek indices.

$$(2) \quad L_M = -\frac{1}{2} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} A_{\mu;\nu} A_{\alpha;\beta},$$

where A_μ is the potential of electromagnetic field and semicolons denote the covariant derivatives with respect to the metric.

Passing to the flat space and developing in series in terms of k , the first-order Lagrangian of interaction between the electromagnetic and gravitational fields has the form:

$$(3) \quad L_{M-G}^{(1)} = -\frac{1}{2} k [A_\mu A_{\nu,\alpha} (h_{\mu\nu,\alpha} + h_{\mu\alpha,\nu} - h_{\nu\alpha,\mu}) + A_{\mu,\nu} A_{\mu,\alpha} h_{\nu\alpha} + A_{\mu,\nu} A_{\alpha,\nu} y_{\mu,\alpha}],$$

where

$$(4) \quad h_{\mu\nu} = y_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} y, \quad y = y_{\alpha\alpha}$$

and commas denote the partial (usual) derivatives.

We shall analyse the process of photon-graviton transformation in external electromagnetic fields. Using the S -matrix formalism, the rules of Feynman type in the external field approximation are deduced, by means of which the corresponding differential cross-sections are obtained [5], [6].

For this type of process, we shall consider three examples of classical fields. The process is described by the Feynman diagram in Fig. 1, where p and $e_\mu^{(a)}(\mathbf{p})$, p' and $e_{\mu\alpha}^{(b)}(\mathbf{p}')$ are the four-momenta and polarization vector or tensor of the initial and final particles, respectively, ($a, b=1, 2$ for real photon and graviton) and q is the four-momentum of the virtual photon.

Taking into account for real gravitons $y=0$ and choosing for photons and graviton the gauge in which $e_4=0$ and $e_{4\alpha}=0$, respectively, the part of (3) Lagrangian casted into the normal form is

$$(5) \quad N[L_{M-G}^{(1)}] = \frac{1}{4} k [A_\mu^{\text{ext}}(x) y_{\nu\alpha,\mu}^{(-)}(x) A_{\nu,\alpha}^{(+)}(x) - A_{\nu,\alpha}^{\text{ext}}(x) y_{\mu\alpha,\nu}^{(-)}(x) A_\mu^{(+)}(x)],$$

where (+) and (-) denote the positive and negative frequency parts, corresponding to the annihilation and creation of particles in x , respectively.

Using the S -matrix formalism we deduce the rules of Feynman type for diagrams in the external electromagnetic field approximation, by which we can calculate the matrix element $\langle p' | S | p \rangle$ on the corresponding approximation. This, for the Coulomb field $\mathbf{E}(x)$ of the nucleus with the charge Ze , it results

$$(6) \quad A_\mu^{\text{ext}}(x) = i \delta_{\mu 4} V(x),$$

where the corresponding potential and field, respectively, are given by

$$(7) \quad V(x) = \frac{Ze}{4\pi |\mathbf{x}|}, \quad \mathbf{E}(x) = -\nabla V(x).$$

Taking into account the Fourier transform of the external electromagnetic potential

$$(8) \quad V(\mathbf{q}) = \frac{1}{3} \int e^{-i\mathbf{q}\cdot\mathbf{x}} V(x) d^3x = \frac{Ze}{3}, \quad \mathbf{q} = \mathbf{p}' - \mathbf{p},$$

$$(2\pi)^2 \quad (2\pi)^2 |\mathbf{q}^2|$$

and the chosen gauge for photons and gravitons, respectively

$$(9) \quad e_4^{(a)}(\mathbf{p}) = e_{4a}^{(b)}(\mathbf{p}') = 0, \quad p_j^1 e_{ij}^{(b)}(\mathbf{p}') = 0.$$

The matrix element in the external field approximation, corresponding to the diagram in Fig. 1, is

$$(10) \quad \langle p' | S | p \rangle = - \frac{ZeK}{8(2\pi)^2 \sqrt{p_0 p_0'}} \int \frac{\delta(q_0)}{|\mathbf{q}|^2} [\delta_{\mu 4} p'_\mu p_\alpha e_{\nu\alpha}^{(a)}(\mathbf{p}) e_\nu^{(b)}(\mathbf{p}) - \\ - \delta_{\nu 4} \delta_{\alpha j} p'_\nu q_j e_{\mu\alpha}^{(b)}(\mathbf{p}') e_\mu^{(a)}(\mathbf{p})] \delta(\mathbf{p}' - \mathbf{p} - \mathbf{q}) d^3 q = F(p, p') \delta(q_0).$$

Finally

$$(11) \quad F(p, p') = - \frac{iZeK}{16(2\pi)^2 K_0^2 \sin^2 \frac{1}{2} \theta} Q(p, p'),$$

where

$$(12) \quad Q(p, p') = p_j e_i^{(a)}(\mathbf{p}) e_{ij}^{(b)}(\mathbf{p}'),$$

θ being the scattering angle (between $\mathbf{p}$ and $\mathbf{p}'$).

The differential cross-section results by averaging over the initial and summing over the final spin (polarization) states of particles:

$$(13) \quad d\sigma = (2\pi)^2 < \sum_{f. sp.} |F(p, p')|^2 >_{i. sp.} p_0^2 d\Omega,$$

where $d\Omega = 2\pi \sin\theta d\theta$ is the infinitesimal solid angle element in the direction of emerging particles.

In order to calculate the $\frac{1}{2} \sum \Omega^2$ expression, the following completeness relations for photons and graviton, respectively, are used:

$$(14) \quad \sum_{a=1}^2 e_i^{(a)}(\mathbf{p}) e_j^{(a)}(\mathbf{p}) = \delta_{ij} - \frac{p_i p_j}{p^2} = d_{ij}(\mathbf{p}), \\ \sum_{b=1}^2 e_{ij}^{(b)}(\mathbf{p}') e_{kl}^{(b)}(\mathbf{p}') = d_{ik}(\mathbf{p}') d_{jl}(\mathbf{p}') + d_{il}(\mathbf{p}') d_{jk}(\mathbf{p}') - d_{ij}(\mathbf{p}') d_{kl}(\mathbf{p}')$$

and the differential cross-section has the form

$$(15) \quad d\sigma = \frac{1}{2} \left[\frac{kZe}{16\pi} \right]^2 (1 - \cos^2 \theta) \cotan^2 \frac{1}{2} \theta d\Omega,$$

in agreement with Mitskevich's result [3], obtained by other way.

For the scattering of photons with p_0 -energy in the homogeneous electrostatic field $\mathbf{E}(\mathbf{x})$ of a plane capacitor, in a domain of $A \times B \times C$ dimensions, the matrix element in the external-field approximation — choosing for photons and graviton the gauge given by (9) — is

$$(16) \quad \langle p' | S | p \rangle = - \frac{ik}{8(2\pi)^2 \sqrt{p_0 p_0'}} \int \delta(q_0) [i \delta_{\mu 4} V(\mathbf{q}) p'_\mu p_\alpha e_{\nu\alpha}^{(a)}(\mathbf{p}) e_{\nu\alpha}^{(b)}(\mathbf{p}') + \\ + \delta_{\nu 4} \delta_{\alpha j} E_j(\mathbf{q}) p'_\nu e_{\mu\alpha}^{(a)}(\mathbf{p}) e_{\mu\alpha}^{(b)}(\mathbf{p}')] \delta(\mathbf{p}' - \mathbf{p} - \mathbf{q}) d^3 q = F(p, p') \delta(q_0),$$

where the relation between the Fourier transforms of the electrostatic potential and

field, respectively

$$(17) \quad q_j V(\mathbf{q}) = i E_j(\mathbf{q}),$$

must be used.

In order to obtain the Fourier transform of the electrostatic field

$$(18) \quad E_j(\mathbf{q}) = -\frac{1}{(2\pi)^2} \int e^{-i\mathbf{q}\cdot\mathbf{x}} E_j(\mathbf{x}) d^3x,$$

we are supposing — unlike [3], [7] — the homogeneous field oriented along the Ox axis, the photons travelling perpendicular to this direction (i.e. along the Oz axis) and that θ is the angle between graviton movement direction and photon one, i.e.

$$(19) \quad \mathbf{E} \equiv (E, 0, 0), \quad \mathbf{p} \equiv (0, 0, p_0), \quad E_j(\mathbf{q}) = \theta_{j1} E(\mathbf{q}),$$

$$\mathbf{p}' \equiv (p_0 \sin\theta \cos\varphi, p_0 \sin\theta \sin\varphi, p_0 \cos\theta),$$

φ being the angle between the $(\mathbf{E}, \mathbf{p})$ and $(\mathbf{p}', \mathbf{p})$ planes. After calculations it results

$$(20) \quad F(p, p') = \frac{2ke}{(2\pi)^2 p_0^3 \sin^2\theta (1 - \cos\theta) \sin\varphi \cos\varphi} \sin\left(\frac{1}{2} A p_0 \sin\theta \cos\varphi\right) \cdot$$

$$\cdot \sin\left(\frac{1}{2} B p_0 \sin\theta \sin\varphi\right) \sin\left[\frac{1}{2} C p_0 (1 - \cos\theta)\right] Q(p, p'),$$

where

$$(21) \quad Q(p, p') = e_i^{(a)}(\mathbf{p}) e_{i1}^{(b)}(\mathbf{p}').$$

Averaging and summing over the polarization states and considering the limit case, when $\theta = \varphi = \pi/2$ the differential cross-section becomes

$$(22) \quad d\sigma = \frac{K^2 E^2 A^2}{2(2\pi)^2 p_0^2} \sin^2\left(\frac{1}{2} B p_0\right) \sin^2\left(\frac{1}{2} C p_0\right) d\Omega,$$

in agreement with Mitskevich [3] and Poznamin's [7] results, obtained in a different manner.

Finally, for the electric dipole with the $\mathbf{d}$ moment, the potential of charges system has the form

$$(23) \quad V(\mathbf{x}) = \frac{\mathbf{d} \cdot \mathbf{x}}{4\pi |\mathbf{x}|^3},$$

$|\mathbf{x}|$ being the distance to this centre. Taking into account the Fourier transform of the charge system's potential — like in (8)

$$(24) \quad V(\mathbf{q}) = -i \frac{d_j q_j}{(2\pi)^3 |\mathbf{q}|^2},$$

and choosing the gauge given by (9), the matrix element in the external field approximation has the form

$$(25) \quad \langle p' | S | p \rangle = \frac{ik}{2(4\pi)^2 \sqrt{p_0 p_0}} \int \frac{\delta(q_0)}{|q|^2} d_j q_j [\delta_{\mu 4} p'_\mu p_\alpha e_\nu^{(a)}(\mathbf{p}) e_{\nu\alpha}^{(b)}(\mathbf{p}') -$$

$$-\delta_{\nu 4} \delta_{\alpha e} p'_\nu q_e e_\mu^{(a)}(\mathbf{p}) e_{\mu\alpha}^{(b)}(\mathbf{p}')] \delta(\mathbf{p}' - \mathbf{p} - \mathbf{q}) d^3 q = F(p, p') \delta(q_0),$$

where again the relation (17) was used. Supposing the electric dipole with the $\mathbf{d}$ moment in the xOz plane the α -angle between $\mathbf{d}$ and the photon movement direction, considered along Oz axis, i.e.

$$(26) \quad \mathbf{d} = (d \sin \alpha, 0, d \cos \alpha), \quad \mathbf{p} = (0, 0, p_0),$$

$$\mathbf{p}' = (p_0 \sin \theta \cos \varphi, p_0 \sin \theta \sin \varphi, p_0 \cos \theta)$$

it results

$$(27) \quad F(p, p') = \frac{kd}{2(4\pi)^2} \left(\cos \alpha - \cotan \frac{1}{2} \theta \cos \varphi \sin \alpha \right) Q(p, p'),$$

with

$$(28) \quad Q(p, p') = e_i^{(a)}(\mathbf{p}) e_{i3}^{(b)}(\mathbf{p}'),$$

θ being the scattering angle and φ — the angle between $(\mathbf{d}, \mathbf{p})$ and $(\mathbf{p}', \mathbf{p})$ planes.

Averaging and summing over the polarization states and considering the direction of photon motion along $\mathbf{d}$ ($\alpha=0$), the differential cross-section becomes

$$(29) \quad d\sigma_{II} = \frac{1}{2} \left[\frac{kd p_0}{16\pi} \right]^2 (1 + \cos^2 \theta) \sin^2 \theta d\Omega$$

and if it is perpendicular on $\mathbf{d}$ ($\alpha=\pi/2$), one obtains

$$(30) \quad d\sigma_{\perp} = \frac{1}{2} \left[\frac{kd p_0}{16\pi} \right]^2 (1 + \cos^2 \theta) (1 + \cos \theta)^2 \cos^2 \varphi d\Omega.$$

This process is not studied in the special literature, but in our opinion it exists, the gravitational wave emission appearing in this case only as a quantum effect.

We conclude finally that unlike the mentioned authors [3], [7], [8], who use the interaction Lagrangian

$$(31) \quad L_M = -\frac{1}{4} \sqrt{-g} g^{\mu\lambda} g^{\nu\delta} F_{\mu\nu} F_{\lambda\delta},$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, we are starting from expression (2)—equivalent at enough small angles with (31) (beside a divergence) — containing the covariant derivatives, according to the principle of minimal coupling, which seems to us naturally.

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University "Al. I. Cuza", Jassy,
Department of Theoretical Physics
and
Technical University "Gh. Asachi", Jassy,
Department of Combustion Engines

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FOTOPRODUCEREA GRAVITONILOR ÎN CÂMPURI
ELECTROMAGNETICE EXTERIOARE

(Rezumat)

Plecând de la principiul cuplajului minimal din teoria cuantică a gravitației și folosind formalismul matricii S , se studiază procesele de fotoproducere a gravitonilor în câmpuri electromagnetice exterioare (câmpul coulombian al nucleului, câmpul electrostatic omogen și câmpul dipolului electric) și se obțin secțiunile diferențiale de difuzie corespunzătoare. Rezultatele sunt în concordanță cu cele din literatura de specialitate, obținute pe o altă cale. Procesul transformării fotonului în graviton în câmpul dipolului electric nu este studiat în literatură, dar după opinia autorilor el există, emisia de unde gravitaționale apărând în acest caz numai ca efect cuantic.

FUZZY MODEL REGARDING THE OPTIMIZATION OF THE PROCESS OF PRECARBONIZATION OF THERMOSTABILIZED PAN FIBRES

BY

D. CORDUNEANU and EMIL LUCA

Abstract. Using a linguistic fuzzy-type model one establishes the optimal values of the parameters characterising the precarbonization phase of thermostabilized PAN fibres for production of performant carbon fibres. For the precursors used, the optimal temperature results 800°C and the optimal stationing time, 148... 262 seconds. The results given by a fuzzy language model relating to optimal temperatures and retention times are in agreement with those obtained by experimental methods.

Key words: fuzzy language model, precarbonization, thermostabilized PAN fibres, optimal temperature.

1. Introduction. Thermostabilized PAN fibres may be exposed to high temperatures of 350...1600°C necessary for the carbonization process, if this treatment is carried out in an inert environment (usually pure nitrogen). After this treatment, which can last for just a few minutes, thermostabilized fibres change their structure essentially, transforming into carbon fibres by eliminating the component heteroatoms under the form of low molecular weight compounds [4]. The nature of gases eliminated during this process is known only on a qualitative basis. A quantitative determination of eliminated gases depending on temperature and time is very difficult to achieve [5].

Unfortunately, during this process heteroatoms are expelled along with carbon atoms (outgoing elements are mainly HCN, CO₂, CO, CH₄), so that a precursor with approximately 68% carbon leads to a fiber weighing only 50% of the initial precursor. Most of the resulting components are eliminated below 1000°C, only intermolecular nitrogen being eliminated, and structural compactions taking place above this

temperature. During a first stage cavities are produced due to the elimination of gases. These cavities are going to compact during a second stage. For this reason, and in order to increase the performances of carbon fibres, the carbonization phase has been divided into two subphases: precarbonization (350...900°C), during which gases are eliminated, and the carbonization itself (1000...1600°C), during which compactations occur. Subsequent treatments to temperatures above 1600°C are usually conducted in argon atmosphere, heated itself above 2800°C, and consist of a very short exposure (under one minute) to these temperatures, in order to increase the Young modulus ("graphitized" fibres) [8]. The mechanisms of reactions occurring during precarbonization can be established indirectly, by analyzing the amounts and the composition of the eliminated gases.

2. Experimental Findings. A PAN precursor cable from the aboriginal production has been used having the following characteristics: number of filaments 10,000; fineness per filament $f_0 = 1.19$ den; breaking stress per filament $r_0 = 7.07$ gf; breaking elongation $a_0 = 13.75\%$; breaking stress $\sigma_0 = 0.643$ GPa.

This cable has been the subject of an oxidative thermostabilization treatment, in one installation in three steps, in air atmosphere, at temperatures of 220°C, 250°C and 265°C, adequate to the three steps, for a total stationing time of 100 minutes. The thermostabilized fibres have the following characteristics: fineness per filament $f_s = 1.09$ den; breaking elongation $a_s = 18.3\%$; breaking stress $\sigma_s = 0.358$ GPa; density $d_s = 1339$ kg/m³, and the following chemical composition: carbon content $C_s = 60.17\%$; nitrogen content $N_s = 20.97\%$; hidrogen content $H_s = 3.57\%$; oxygen content $O_s = 15.29\%$ [1].

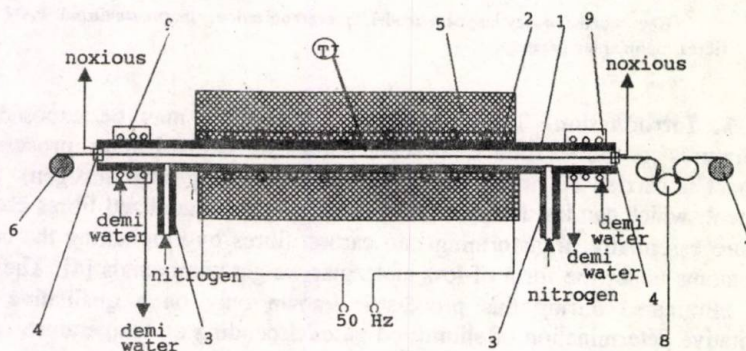


Fig. 1 — The precarbonization furnace: 1 — metallic tube; 2 — electrical resistance; 3 — tubs; 4 — tights; 5 — thermic isolation; 6 — spooler cheese; 7 — spooler; 8 — group of waves; 9 — cooling system.

The precarbonization treatments of the cable have been carried out in an installation (Fig. 1), by modifying the temperatures in the range of 350...850°C, and for each prescribed temperature the stationing time has been modified in the range of

1.2 ... 5.5 minutes.

40 precarbonization fibre samples, labeled $P_1 \dots P_{40}$, have been obtained from a single thermostabilized fibre labeled P_1 . The labels of these samples and the conditions they were obtained are depicted in Table 1.

Table 1
Parameters of precarbonization of sample P_1 .

No.	Sample code	Temperature °C	Time sec	Speed of heating °C/min	No.	Sample code	Temperature °C	Time sec	Speed of heating °C/min
1	P_1	328	66	4.97	21	P_{21}	700	150	4.67
2	P_2	329	75	4.39	22	P_{22}	695	200	3.48
3	P_3	330	85	3.88	23	P_{23}	705	260	2.71
4	P_4	330	120	2.75	24	P_{24}	700	325	2.15
5	P_5	330	150	2.20	25	P_{25}	760	66	11.52
6	P_6	328	200	1.64	26	P_{26}	762	75	10.16
7	P_7	332	260	1.28	27	P_{27}	760	85	8.94
8	P_8	330	325	1.02	28	P_{28}	758	120	6.32
9	P_9	628	66	9.52	29	P_{29}	760	150	5.07
10	P_{10}	630	75	8.40	30	P_{30}	758	200	3.79
11	P_{11}	631	85	7.42	31	P_{31}	758	260	2.92
12	P_{12}	630	120	5.25	32	P_{32}	761	325	2.34
13	P_{13}	632	150	4.21	33	P_{33}	830	66	12.58
14	P_{14}	629	200	3.15	34	P_{34}	831	75	11.08
15	P_{15}	630	260	2.42	35	P_{35}	829	75	9.75
16	P_{16}	630	325	1.94	36	P_{36}	828	120	6.90
17	P_{17}	700	66	10.61	37	P_{37}	830	150	5.53
18	P_{18}	705	75	9.40	38	P_{38}	830	200	4.15
19	P_{19}	698	85	8.21	39	P_{39}	831	260	3.20
20	P_{20}	700	120	5.83	40	P_{40}	832	325	2.56

Table 2 shows the physico-mechanical characteristics of samples precarbonized in the conditions in Table 1 and Table 3 shows the chemical characteristics of the

same samples.

Table 2

Physico-mechanical characteristics of precarbonized fibres

No.	Sam- ple label	Fine ness f_p den	Density d_p kg/m ³	Breaking stress r_p gf	Breaking elon- gation a_p %	Tenac- ity gf/den	Breaking stress σ_p GPa	Young modulus E_p GPa
1	P_1	1.088	1362.1	2.75	3.15	2.5276	0.2794	8.8692
2	P_2	1.09	1363.3	2.68	3.5	2.4587	0.2715	7.7574
3	P_3	1.09	1367.2	2.61	4.07	2.3945	0.2652	6.5153
4	P_4	1.08	1379.4	2.45	4.72	2.2685	0.2558	5.4197
5	P_5	1.03	1380.1	2.35	4.7	2.2816	0.2699	5.7427
6	P_6	1.01	1364.8	1.18	3.53	1.1683	0.1394	3.9486
7	P_7	1	1364.4	0.89	3.35	0.8900	0.1072	3.2004
8	P_8	0.98	1375	0.76	3.06	0.7755	0.0961	3.1395
9	P_9	0.83	1514.75	6.31	3.33	7.6024	1.2250	36.7959
10	P_{10}	0.83	1512.57	3.97	4.78	4.7831	0.7696	16.1003
11	P_{11}	0.854	1500	3.28	4.79	3.8407	0.5956	12.4344
12	P_{12}	0.86	1500.66	3.04	3.47	3.5349	0.5446	15.6943
13	P_{13}	0.86	1497	3.76	3.03	4.3721	0.6719	22.1759
14	P_{14}	0.85	1503.15	3.85	2.55	4.5194	0.7072	27.7330
15	P_{15}	0.846	1519.4	4.62	2.63	5.4610	0.8659	32.9253
16	P_{16}	0.86	1523.65	5.11	2.6	5.4919	0.9294	35.7476
17	P_{17}	0.84	1538.472	5.92	2.31	7.0476	1.1396	49.3347
18	P_{18}	0.82	1539.7	5.06	2.53	6.1707	1.0230	40.4343
19	P_{19}	0.84	1538.4	4.43	2.57	5.2738	0.8528	33.1813
20	P_{20}	0.842	1538.11	4.39	2.77	5.2138	0.8409	30.3570
21	P_{21}	0.86	1542.8	4.27	2.28	4.9651	0.7864	34.4919
22	P_{22}	0.86	1556	3.6	1.65	4.1860	0.6687	40.5268
23	P_{23}	0.86	1562	3.28	1.43	3.8140	0.6116	42.7694
24	P_{24}	0.87	1563	2.96	1.28	3.4023	0.5397	42.1613

Table 2 (continued)

25	P_{25}	0.82	1612.82	5.59	1.97	6.8171	1.1838	60.0918
26	P_{26}	0.82	1604.3	4.18	1.956	5.0976	0.8805	45.0170
27	P_{27}	0.808	1603.3	4.19	2.233	5.1856	0.9085	40.6845
28	P_{28}	0.82	1612.54	3.68	2.43	4.4878	0.7792	32.0653
29	P_{29}	0.81	1616.8	4.35	2.19	5.3704	0.9464	43.2158
30	P_{30}	0.807	1623.58	5.48	1.61	6.7906	1.2062	74.9191
31	P_{31}	0.8	1635.58	5.51	1.53	6.8875	1.2432	81.2575
32	P_{32}	0.82	1640.24	6.15	1.41	7.5000	1.3245	93.9393
33	P_{33}	0.65	1612	4.48	2.07	6.8923	1.5091	72.9050
34	P_{34}	0.66	1613.18	5.03	2.24	7.6212	1.6447	73.4220
35	P_{35}	0.69	1630.32	5.16	2.33	7.4783	1.5600	66.9545
36	P_{36}	0.724	1638.17	4.98	2.62	6.8785	1.3741	52.4471
37	P_{37}	0.748	1645.32	4.77	2.38	6.3770	1.2384	52.0355
38	P_{38}	0.78	1682.46	4.745	1.83	6.0833	1.1585	63.3071
39	P_{39}	0.797	1714	4.53	1.68	5.6838	1.0792	64.2384
40	P_{40}	0.79	1723	4.347	1.4	5.5025	1.0596	75.6840
41	PS_1	1.09	1339	3.27	1.69	3.0000	0.3254	19.2531
	P_0	1.19	1057	7.62	8.45	6.4034	0.5022	5.9428

3. Determination of Optimal Parameters for Precarbonization. Different precarbonized fibres are obtained, bearing significantly different characteristics, from a single thermostabilized fibre, by varying the two working parameters, namely the temperature and the stationing time. A deterministic model is very difficult to establish, because of the complex processes determined by the evolution of chemical reaction rates.

This is why a linguistic (fuzzy) model has been used in the optimization of the process, that is, in determining the best working conditions [11]. This model assumes that the precarbonization process is a non-linear finite state system, a "black box" type of system, with measurements being conducted on its inputs and outputs [2].

The characteristics of the thermostabilized fibre, namely the breaking stress and the oxygen content, which were considered outputs of the fuzzy model of thermo-

stabilization, are now considered inputs [1]. The temperatures and the stationing times are system parameters, and the characteristics of the precarbonization fibres, namely density, breaking stress and Young modulus per filament, are considered outputs [3]. The scheme of the model is depicted in Fig. 2.

Table 3
Chemical characteristics of precarbonized fibres

Sample label	Carbon C_p %	Nitro- gen N_p %	Hidro- gen H_p %	Oxy- gen O_p %	Sample label	Carbon C_p %	Nitro- gen N_p %	Hidro- gen H_p %	Oxy- gen O_p %
P_1	61.4	21.2	3.534	13.87	P_{21}	71.57	19.4	2	7.29
P_2	61.68	20.9	3.579	13.84	P_{22}	71.8	19	1.94	7.58
P_3	61.71	20.8	3.54	13.95	P_{23}	71.95	19.16	1.67	7.22
P_4	61.52	20.64	3.4	14.44	P_{24}	72.7	19.28	1.54	6.48
P_5	61.92	20.22	3.38	14.48	P_{25}	66.77	17.2	1.5	14.53
P_6	65.59	21.76	4.18	8.47	P_{26}	67.65	17.258	1.5	13.59
P_7	66.2	22.4	4.222	7.18	P_{27}	68.81	17.3	1.546	12.11
P_8	66.13	22.67	4.149	7.05	P_{28}	72.48	17.867	1.531	8.12
P_9	71.09	19.79	2.28	6.84	P_{29}	72.4	17.92	1.41	8.27
P_{10}	69.39	19.96	2.6	8.05	P_{30}	74.55	17.98	1.377	6.09
P_{11}	67.7	19.7	2.57	10.03	P_{31}	75.66	18.129	1.39	4.82
P_{12}	68.52	19.48	2.23	9.77	P_{32}	73.17	17.9	1.31	7.83
P_{13}	70.8	19.47	2.13	7.60	P_{33}	70.6	17.55	1.251	10.60
P_{14}	71.47	19.31	2.16	7.06	P_{34}	72.2	18.027	1.37	8.40
P_{15}	69.2	18.85	2.15	9.80	P_{35}	73.85	18.1	1.373	6.67
P_{16}	65.86	18.13	2.1	13.91	P_{36}	74.8	18.24	1.43	5.53
P_{17}	68.54	18.83	2.09	10.52	P_{37}	74.3	17.97	1.33	6.40
P_{18}	69.98	19.467	2.13	8.42	P_{38}	74.8	17.381	1.028	6.79
P_{19}	71.33	19.35	2.12	7.20	P_{39}	76.56	17.48	0.943	5.02
P_{20}	71.03	19.08	1.99	7.90	P_{40}	79.62	18.42	1.02	0.94
					PS_1	60.17	20.97	3.57	15.29
					P_0	66.74	24.48	5.75	3.03

The procedures of fuzzyfication and defuzzyfication of the crisp variables of the system are those which involve the use of the triangle rule, that is the following rule (1) is used for fuzzyfication and the rule (2) for defuzzyfication [3], [8]:

$$(1) \quad \mu(x) = 1 - \frac{|x - x_0|}{h},$$

$$(2) \quad |x - x_0| = (1 - \mu(x)) h.$$

The language we introduce is given by

$$(3) \quad \langle \text{small, medium, great} \rangle.$$

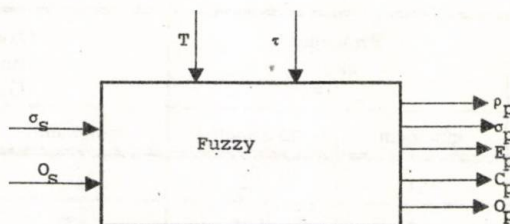


Fig. 2 — The scheme of the fuzzy model of precarbonization.

3.1. Fuzzyfication of Inputs. The crisp values for the inputs are considered to be $\sigma_{r0} = 0.358$ GPa and $O_0 = 15.29\%$ [1]. The limit values for establishing the membership functions are shown in Table 4.

Consequently the fuzzy matrix for inputs is

$$(4) \quad X = \begin{bmatrix} M/0.04 \\ mM/0.947 \end{bmatrix},$$

where the first element represents the attribute and its confidence degree for the breaking stress, and the second element represents the same for the oxygen content.

3.2. Fuzzyfication of Parameters. The same exposure temperature and the same stationing time have been used for the precarbonization treatment of each sample.

Temperatures T_p below 500°C were considered small because a large number of heteroatoms still remain in the structure, and temperatures above 800°C were considered great because the resultant gases are expelled at a fast rate, due to the fast heating rate, and this leads to structural defects. Also, for an average heating rate, stationing times τ_p shorter than 130 seconds were considered small, and times longer than 280 seconds were considered great.

The limit value for the fuzzyfication of system parameters are given in Table

5.

3.3. *Fuzzyfication of Outputs.* a) *The density* of precarbonization fibers is proportional to the carbon content of the fibre and inverse proportional to the number of cavities and structural defects. Densities under 1500 kg/m^3 were considered small and densities over 1600 kg/m^3 were considered great.

Table 4

Limit values of inputs for establishing the membership functions

Attribute	Breacking stress σ_s GPa		Oxygen content O_s %	
	minimum	maximum	minimum	maximum
small (<i>m</i>)	0.1	0.2	8	14.15
medium (<i>mM</i>)	0.2	0.35	14.5	16
great (<i>M</i>)	0.35	0.55	16	20

Table 5

Limit values of parameters for establishing the membership functions

Attribute	Temperature, T_p °C		Stationing time, τ_p seconds	
	minimum	maximum	minimum	maximum
small (<i>m</i>)	300	500	60	130
medium (<i>mM</i>)	500	801	130	280
great (<i>M</i>)	801	900	280	350

b) *The breaking stress per filament* is given by the links of macromolecular structures directed along the axis of the fibre. The compact structures consisting of carbon aromatic cycles will lead to an increased breaking stress. For precarbonized fibres, breaking stress values were considered small under 0.55 GPa and great above 1 GPa.

c) *The Young modulus per filament* is not very great after precarbonization, due on one hand to the still large number of heteroatoms, and on the other hand to the large number of structural defects remaining after the departure of gases, which decreases due to recombination of chains. For precarbonization fibres, values under 45 GPa were considered small and values above 65 GPa were considered great.

d) *The carbon content* is inverse proportional to the amount of gases expelled during the process. The percentage is small below 45% and great above 65%.

e) *The oxygen content* in the fibre has very low values depending on the existing chemical structure, and it is given by the elements which still contain oxygen atoms and which have not yet been eliminated. Precarbonization fibres have a small oxygen content if it is under 8% and a great one if it is above 12%.

Table 6 shows the limit values for outputs evaluated in order to carry out the fuzzification.

Table 6

Limit values of outputs for establishing the membership functions

Attribute		small (<i>m</i>)	medium (<i>mM</i>)	great (<i>M</i>)
Density kg/m ₃	minimum	1300	1500	1600
	maximum	1500	1600	1750
Breacking stress, GPa	minimum	0.15	0.55	1
	maximum	0.55	1	2
Young modulus, GPa	minimum	5	45	65
	maximum	45	65	110
Carbon content, %	minimum	60	65	75
	maximum	65	70	80
Oxygen content, %	minimum	0.5	8	12
	maximum	8	12	18

3.4. Description of the Fuzzy Model. The relations giving the values of the membership functions are given below, namely (5) for small values, (6) for medium values and (7) for great values [5], [6], [10]:

$$(5) \quad \mu_m = 1 - \frac{|x - x_0|}{x_2 - x_1},$$

$$(6) \quad \mu_{mM} = 1 - \frac{|x - (x_2 + x_3)/2|}{(x_3 - x_2)/2},$$

$$(7) \quad \mu_M = 1 - \frac{|x - x_4|}{x_4 - x_3}.$$

By applying these rules to crisp values in Tables 1 and 2, the fuzzy following matrixes (S_s, S_m, S_g) (8) for parameters and (Y_s, Y_m, Y_g) (9), (10), (11) are obtained:

(8) $S_m =$	0.8600	0.9143	$S_{mM} =$	0.000	0.000	$S_M =$	0.000	0.000
	0.8550	0.7857		0.000	0.000		0.000	0.000
	0.8500	0.6429		0.000	0.000		0.000	0.000
	0.8500	0.1429		0.000	0.000		0.000	0.000
	0.8500	0.0000		0.000	0.267		0.000	0.000
	0.8600	0.0000		0.000	0.933		0.000	0.000
	0.8400	0.0000		0.000	0.267		0.000	0.000
	0.8500	0.0000		0.000	0.000		0.000	0.643
	0.0000	0.9143		0.850	0.000		0.000	0.000
	0.0000	0.7857		0.864	0.000		0.000	0.000
	0.0000	0.6429		0.870	0.000		0.000	0.000
	0.0000	0.1429		0.864	0.000		0.000	0.000
	0.0000	0.0000		0.877	0.267		0.000	0.000
	0.0000	0.0000		0.857	0.933		0.000	0.000
	0.0000	0.0000		0.864	0.267		0.000	0.000
	0.0000	0.0000		0.864	0.000		0.000	0.643
	0.0000	0.9143		0.671	0.000		0.000	0.000
	0.0000	0.7857		0.638	0.000		0.000	0.000
	0.0000	0.6429		0.684	0.000		0.000	0.000
	0.0000	0.1429		0.671	0.000		0.000	0.000
	0.0000	0.0000		0.671	0.267		0.000	0.000
	0.0000	0.0000		0.704	0.933		0.000	0.000
	0.0000	0.0000		0.638	0.267		0.000	0.000
	0.0000	0.0000		0.671	0.000		0.000	0.643
	0.0000	0.9143		0.272	0.000		0.000	0.000
	0.0000	0.7857		0.259	0.000		0.000	0.000
	0.0000	0.6429		0.272	0.000		0.000	0.000
	0.0000	0.1429		0.286	0.000		0.000	0.000
	0.0000	0.0000		0.272	0.267		0.000	0.000
	0.0000	0.0000		0.286	0.933		0.000	0.000
	0.0000	0.0000		0.286	0.267		0.000	0.000
	0.0000	0.0000		0.266	0.000		0.000	0.643
	0.0000	0.9143		0.000	0.000		0.000	0.000
	0.0000	0.7857		0.000	0.000		0.000	0.000
	0.0000	0.6429		0.000	0.000		0.000	0.000
	0.0000	0.1429		0.000	0.000		0.000	0.000
	0.0000	0.0000		0.000	0.267		0.000	0.000
	0.0000	0.0000		0.000	0.933		0.000	0.000
	0.0000	0.0000		0.000	0.267		0.000	0.000
	0.0000	0.0000		0.000	0.000		0.000	0.643

(9) $Y_m =$

0.690	0.677	0.903	0.720	0.000
0.684	0.696	0.931	0.664	0.000
0.664	0.712	0.962	0.658	0.000
0.603	0.735	0.990	0.696	0.000
0.600	0.700	0.981	0.616	0.000
0.676	0.973	0.974	0.000	0.000
0.678	0.893	0.955	0.000	0.110
0.625	0.865	0.953	0.000	0.127
0.000	0.000	0.205	0.000	0.155
0.000	0.000	0.722	0.000	0.000
0.000	0.000	0.814	0.000	0.000
0.000	0.014	0.733	0.000	0.000
0.015	0.000	0.571	0.000	0.053
0.000	0.000	0.432	0.000	0.125
0.000	0.000	0.302	0.000	0.000
0.000	0.000	0.231	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.114	0.000	0.000
0.000	0.000	0.295	0.000	0.107
0.000	0.000	0.366	0.000	0.013
0.000	0.000	0.263	0.000	0.095
0.000	0.000	0.112	0.000	0.056
0.000	0.000	0.056	0.000	0.104
0.000	0.026	0.071	0.000	0.203
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.108	0.000	0.000
0.000	0.000	0.323	0.000	0.000
0.000	0.000	0.045	0.000	0.000
0.000	0.000	0.000	0.000	0.254
0.000	0.000	0.000	0.000	0.424
0.000	0.000	0.000	0.000	0.023
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.178
0.000	0.000	0.000	0.000	0.329
0.000	0.000	0.000	0.000	0.213
0.000	0.000	0.000	0.000	0.161
0.000	0.000	0.000	0.000	0.398
0.000	0.000	0.000	0.000	0.941

(10) $Y_{mM} =$

0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.236	0.235
0.000	0.000	0.000	0.480	0.000
0.000	0.000	0.000	0.452	0.000
0.295	0.000	0.000	0.000	0.000
0.251	0.976	0.000	0.244	0.025
0.000	0.203	0.000	0.920	0.985
0.013	0.000	0.000	0.592	0.885
0.000	0.542	0.000	0.000	0.000
0.063	0.699	0.000	0.000	0.000
0.388	0.596	0.000	0.320	0.900
0.473	0.314	0.000	0.344	0.000
0.769	0.000	0.433	0.584	0.730
0.794	0.000	0.000	0.008	0.211
0.768	0.654	0.000	0.000	0.000
0.762	0.707	0.000	0.000	0.000
0.856	0.949	0.000	0.000	0.000
0.880	0.528	0.000	0.000	0.000
0.760	0.274	0.000	0.000	0.000
0.740	0.000	0.000	0.000	0.000
0.000	0.000	0.491	0.708	0.000
0.000	0.531	0.002	0.940	0.000
0.000	0.407	0.000	0.476	0.000
0.000	0.981	0.000	0.000	0.061
0.000	0.238	0.000	0.000	0.135
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.700
0.000	0.000	0.000	0.000	0.201
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.745	0.000	0.000
0.000	0.000	0.704	0.000	0.000
0.000	0.000	0.169	0.000	0.000
0.000	0.000	0.076	0.000	0.000
0.000	0.000	0.000	0.000	0.000

(11)

 $Y_M =$

0.000	0.000	0.000	0.000	0.311
0.000	0.000	0.000	0.000	0.307
0.000	0.000	0.000	0.000	0.325
0.000	0.000	0.000	0.00	0.407
0.000	0.000	0.000	0.000	0.413
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.225	0.000	0.109	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.080	0.000
0.000	0.000	0.000	0.147	0.000
0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.318
0.000	0.140	0.000	0.000	0.000
0.000	0.023	0.000	0.000	0.000
0.000	0.000	0.000	0.133	0.000
0.000	0.000	0.000	0.103	0.000
0.000	0.000	0.000	0.157	0.000
0.000	0.000	0.000	0.148	0.000
0.00	0.000	0.000	0.195	0.000
0.000	0.000	0.000	0.270	0.000
0.085	0.184	0.000	0.000	0.422
0.029	0.000	0.000	0.000	0.265
0.022	0.000	0.000	0.000	0.019
0.084	0.000	0.000	0.248	0.000
0.112	0.000	0.000	0.240	0.000
0.157	0.206	0.220	0.455	0.000
0.237	0.243	0.361	0.566	0.000
0.268	0.325	0.643	0.317	0.000
0.080	0.509	0.176	0.060	0.000
0.088	0.645	0.187	0.220	0.000
0.202	0.560	0.043	0.385	0.000
0.254	0.374	0.000	0.480	0.000
0.302	0.238	0.000	0.430	0.000
0.550	0.159	0.000	0.480	0.000
0.760	0.079	0.000	0.656	0.000
0.820	0.060	0.237	0.962	0.000

The linguistic system thus constituted is composed of linguistic rules of the form:

- (12) "If the precursor has the breaking stress $\{s, m, g\}$ and the oxygen content $\{s, m, g\}$ and if it is precarbonized at the temperature $\{s, m, g\}$ and the stationing time $\{s, m, g\}$, then the fibre has the density $\{s, m, g\}$, the breacking stress per filament $\{s, m, g\}$, the Young modulus $\{s, m, g\}$, the carbon content $\{s, m, g\}$ and the oxygen content $\{s, m, g\}$ ".

Rule (12) may be rewritten symbolically as follows:

- (13) "If (σ_s IS $\{m, mM, M\}$ BELIEF A_1) AND (O_s IS $\{m, mM, M\}$ BELIEF A_2) AND (T_p IS $\{m, mM, M\}$ BELIEF A_3) AND (τ_p IS $\{m, mM, M\}$ BELIEF A_4) THEN (ρ_p IS $\{m, mM, M\}$ BELIEF A_5) AND (σ_p IS $\{m, mM, M\}$ BELIEF A_6) AND (E_p IS $\{m, mM, M\}$ BELIEF A_7) AND (C_p IS $\{m, mM, M\}$ BELIEF A_8) AND (O_p IS $\{m, mM, M\}$ BELIEF A_9)".

In relation (13), symbols $A_1 \dots A_9$ are the degree of confidence given to one attribute in the set $\{m, mM, M\}$.

Fuzzy matrix (9)...(11) may be grouped together within a single table containing for each variable the attribute and the degree of confidence in each cell (Table 7).

The logical relation (13) is iterated 40 times for each line in Table 7.

The logical system (13) derived by the interpretation of Table 7 is described by

- (14) "If (σ_s IS M BELIEF 0.04) AND (O_s IS mM BELIEF 0.947) AND (T_{pi} IS A_{i1} BELIEF B_{i1}) AND (τ_{pi} IS A_{i2} BELIEF B_{i2}) THEN (ρ_{pi} IS A_{i3} BELIEF B_{i3}) AND (σ_{pi} IS A_{i4} BELIEF B_{i4}) AND (E_{pi} IS A_{i5} BELIEF B_{i5}) AND (C_{pi} IS A_{i6} BELIEF B_{i6}) AND (O_{pi} IS A_{i7} BELIEF B_{i7})".

In relation (14), symbols $A_1 \dots A_7$ are the attributes for the variables described, and $B_1 \dots B_7$ are the degrees of confidence for the mentioned attributes. The index i takes values in the range 1...40, so that the system (14) contains 40 logical relations.

If the following restrictions are enforced upon the system: medium density, medium breacking stress, small Young modulus, great carbon content and small oxygen content, then, by automatically extracting those instructions which fulfill six conditions, Table 7 is reduced to the only six lines 14, 19, 20, 21, 22 and 23, the ones which have the attributes mM , mM , m , M and m in columns respectively, and the system (14) is reduced to only three equations (Table 8).

If a new restriction is introduced in which the oxygen content has the attribute small with the high degree of confidence, then what is left is the logical instruction (14) which gives:

- (15) "If (σ_s IS M BELIEF 0.04) AND (O_s IS mM BELIEF 0.947) AND (T_p IS mM BELIEF 0.857) AND (τ_p IS mM BELIEF 0.933) THEN (ρ_p IS mM BELIEF 0.063) AND (σ_p IS M BELIEF 0.699) AND (E_p IS mM BELIEF 0.432) AND (C_p IS M BELIEF 0.147) AND (O_p IS m BELIEF 0.125)".

Table 7

The content of the logical relations forming the fuzzy system

No.	Parameters			Outputs			
	Temperature <i>T</i>	Time <i>t</i>	Density <i>d</i>	Breacking stress <i>s</i>	Young modulus <i>E</i>	Carbon content <i>C</i>	Oxygen content <i>O</i>
1	<i>m</i> /0.8600	<i>m</i> /0.9143	<i>m</i> /0.690	<i>m</i> /0.677	<i>m</i> /0.903	<i>m</i> /0.720	<i>M</i> /0.311
2	<i>m</i> /0.8550	<i>m</i> /0.7857	<i>m</i> /0.684	<i>m</i> /0.696	<i>m</i> /0.931	<i>m</i> /0.664	<i>M</i> /0.307
3	<i>m</i> /0.8550	<i>m</i> /0.6429	<i>m</i> /0.664	<i>m</i> /0.712	<i>m</i> /0.962	<i>m</i> /0.658	<i>M</i> /0.325
4	<i>m</i> /0.8550	<i>m</i> /0.1429	<i>m</i> /0.603	<i>m</i> /0.735	<i>m</i> /0.990	<i>m</i> /0.696	<i>M</i> /0.407
5	<i>m</i> /0.8550	<i>mM</i> /0.267	<i>m</i> /0.600	<i>m</i> /0.700	<i>m</i> /0.981	<i>m</i> /0.616	<i>M</i> /0.413
6	<i>m</i> /0.8600	<i>mM</i> /0.933	<i>m</i> /0.676	<i>m</i> /0.973	<i>m</i> /0.974	<i>mM</i> /0.236	<i>mM</i> /0.235
7	<i>m</i> /0.8400	<i>mM</i> /0.267	<i>m</i> /0.678	<i>m</i> /0.893	<i>m</i> /0.955	<i>mM</i> /0.480	<i>m</i> /0.110
8	<i>m</i> /0.8500	<i>M</i> /0.643	<i>m</i> /0.625	<i>m</i> /0.865	<i>m</i> /0.953	<i>mM</i> /0.452	<i>m</i> /0.127
9	<i>mM</i> /0.850	<i>m</i> /0.9143	<i>mM</i> /0.295	<i>M</i> /0.225	<i>m</i> /0.205	<i>M</i> /0.109	<i>m</i> /0.155
10	<i>mM</i> /0.864	<i>m</i> /0.7857	<i>mM</i> /0.251	<i>mM</i> /0.976	<i>m</i> /0.722	<i>mM</i> /0.244	<i>mM</i> /0.025
11	<i>mM</i> /0.870	<i>m</i> /0.6429	<i>mM</i> /0.001	<i>mM</i> /0.203	<i>m</i> /0.814	<i>mM</i> /0.920	<i>mM</i> /0.985
12	<i>mM</i> /0.864	<i>m</i> /0.1429	<i>mM</i> /0.013	<i>m</i> /0.014	<i>m</i> /0.733	<i>mM</i> /0.592	<i>mM</i> /0.885
13	<i>mM</i> /0.877	<i>mM</i> /0.267	<i>m</i> /0.015	<i>mM</i> /0.542	<i>m</i> /0.571	<i>M</i> /0.320	<i>m</i> /0.053
14	<i>mM</i> /0.857	<i>mM</i> /0.933	<i>mM</i> /0.063	<i>mM</i> /0.699	<i>m</i> /0.432	<i>M</i> /0.147	<i>m</i> /0.125
15	<i>mM</i> /0.864	<i>mM</i> /0.267	<i>mM</i> /0.338	<i>mM</i> /0.596	<i>m</i> /0.302	<i>mM</i> /0.320	<i>mM</i> /0.900
16	<i>mM</i> /0.864	<i>M</i> /0.643	<i>mM</i> /0.473	<i>mM</i> /0.314	<i>m</i> /0.231	<i>mM</i> /0.344	<i>M</i> /0.318
17	<i>mM</i> /0.671	<i>m</i> /0.9143	<i>mM</i> /0.769	<i>M</i> /0.140	<i>mM</i> /0.433	<i>mM</i> /0.584	<i>mM</i> /0.730
18	<i>mM</i> /0.638	<i>m</i> /0.7857	<i>mM</i> /0.794	<i>M</i> /0.023	<i>m</i> /0.114	<i>mM</i> /0.008	<i>mM</i> /0.211
19	<i>mM</i> /0.684	<i>m</i> /0.6429	<i>mM</i> /0.768	<i>mM</i> /0.654	<i>m</i> /0.295	<i>M</i> /0.133	<i>m</i> /0.107
20	<i>mM</i> /0.671	<i>m</i> /0.1429	<i>mM</i> /0.762	<i>mM</i> /0.707	<i>m</i> /0.366	<i>M</i> /0.103	<i>m</i> /0.013
21	<i>mM</i> /0.671	<i>mM</i> /0.267	<i>mM</i> /0.856	<i>mM</i> /0.949	<i>m</i> /0.263	<i>M</i> /0.157	<i>m</i> /0.095
22	<i>mM</i> /0.704	<i>mM</i> /0.933	<i>mM</i> /0.880	<i>mM</i> /0.528	<i>m</i> /0.112	<i>M</i> /0.148	<i>m</i> /0.056
23	<i>mM</i> /0.638	<i>mM</i> /0.267	<i>mM</i> /0.760	<i>mM</i> /0.274	<i>m</i> /0.056	<i>M</i> /0.195	<i>m</i> /0.104
24	<i>mM</i> /0.671	<i>M</i> /0.643	<i>mM</i> /0.740	<i>m</i> /0.026	<i>m</i> /0.071	<i>M</i> /0.270	<i>m</i> /0.203

Table 7 (continued)

25	mM/0.272	m/0.9143	M/0.085	M/0.184	mM/0.491	mM/0.708	M/0.422
26	mM/0.259	m/0.7857	M/0.029	mM/0.531	mM/0.002	mM/0.940	M/0.265
27	mM/0.272	m/0.6429	M/0.022	mM/0.407	m/0.108	mM/0.476	M/0.019
28	mM/0.286	m/0.1429	M/0.084	M/0.981	m/0.323	M/0.248	mM/0.061
29	mM/0.272	mM/0.267	M/0.112	M/0.238	m/0.045	M/0.240	mM/0.135
30	mM/0.286	mM/0.933	M/0.157	M/0.206	M/0.220	M/0.455	m/0.254
31	mM/0.266	mM/0.267	M/0.237	M/0.243	M/0.361	M/0.566	m/0.424
32	mM/0.266	M/0.643	M/0.268	M/0.325	M/0.643	M/0.317	m/0.023
33	M/0.293	m/0.9143	M/0.080	M/0.509	M/0.176	M/0.060	mM/0.700
34	M/0.303	m/0.7857	M/0.088	M/0.645	M/0.187	M/0.220	mM/0.201
35	M/0.283	m/0.6429	M/0.202	M/0.560	M/0.043	M/0.385	m/0.178
36	M/0.273	m/0.1429	M/0.254	M/0.374	mM/0.745	M/0.480	m/0.329
37	M/0.293	mM/0.267	M/0.302	M/0.238	mM/0.704	M/0.430	m/0.213
38	M/0.293	mM/0.933	M/0.550	M/0.159	mM/0.169	M/0.480	m/0.161
39	M/0.303	mM/0.267	M/0.760	M/0.079	mM/0.076	M/0.656	m/0.398
40	M/0.313	M/0.643	M/0.820	M/0.060	M/0.237	M/0.942	m/0.941

Table 8

Logical instructions in system (14) after restrictions have been enforced

No.	Parameters			Outputs			
	Tempera- ture <i>T</i>	Time <i>t</i>	Density <i>d</i>	Breaking stress <i>s</i>	Young modulus <i>E</i>	Carbon content <i>C</i>	Oxygen content <i>O</i>
14	mM/0.857	mM/0.933	mM/0.063	mM/0.699	m/0.432	M/0.147	m/0.125
19	mM/0.684	m/0.6429	mM/0.768	mM/0.654	m/0.295	M/0.133	m/0.107
20	mM/0.671	m/0.1429	mM/0.762	mM/0.707	m/0.366	M/0.103	m/0.013
21	mM/0.671	mM/0.267	mM/0.856	mM/0.949	m/0.263	M/0.157	m/0.095
22	mM/0.704	mM/0.933	mM/0.880	mM/0.528	m/0.122	M/0.148	m/0.056
23	mM/0.638	mM/0.267	mM/0.760	mM/0.274	m/0.056	M/0.185	m/0.104

4. Interpretation of the Results. Crisp values in Table 8 are obtained by defuzzification of variables given in relation (15), making use of (2) as the following triangle rule [8]:

- (16) "If a thermostabilized precursor with a breacking stress of 0.343 GPa and an oxygen content of 15.56%, is precarbonized at the temperature of 629°C, for a stationing time of 200 sec, then a precarbonized fiber is obtained with a density of 1503.15 kg/m³, a breacking stress of 0.7072 GPa, a Young modulus of 27.733 GPa, and which has the following chemical composition: carbon content 71.47% and oxygen content 7.07%".

The logical relation (6) once defuzzified, becomes Table 9.

Table 9
Crisp values for relation (14)

Inputs	σ_p GPa	0.343
	O_p %	15.56
Parameters	T_p °C	629
	τ_p sec	200
Outputs	d_p kg/m ³	1503.15
	σ_p GPa	0.7072
	E_p GPa	27.733
	C_p %	71.47
	O_p %	7.07

Relation (16) shows that in order to obtain a precarbonized fibre with optimal characteristics, a stationing time of 200 seconds and a temperature of exposure of 629°C are needed [7]. Table 10 gives the results obtained by precarbonizing two samples of fibres (PAN precursor cable from the aboriginal production code P_{14} and PAN precursor cable type SAF Couraulds-England), in conditions found by fuzzy optimization.

Table 10 shows that the characteristics of precarbonized fibres are close for both types of fibres. Comparing the characteristics of precarbonized fibres obtained from precursor to those of precarbonized fibres obtained from precursor SAF, insignificant differences are observed.

The fibres obtained from precursor SAF have density, elastic modulus and carbon content smaller, that those obtained from indigen precursors.

There are insignificant differences for fibres resulted from the SAF precursor, whose characteristics are bigger. For the same type of precursor we can notice that

the values of the characteristics are bigger at temperatures above 629°C and a stationing time of 200 seconds, and this is why they are considered to be optimal values.

Table 10
Results of precarbonization in optimal conditions

		Indigene PAN fibres	SAF PAN fibers
Parameters	T_p °C	629	629
	τ_p sec	200	200
Results of analysis	d_p kg/m ³	1503.15	1658
	σ_p GPa	0.707	0.83
	E_p GPa	27.733	32.18
	C_p %	71.47	76.21
	O_p %	7.07	6.82

5. Conclusions. 1. The use of a linguistic fuzzy-type model allowed us to determine optimal values for the parameters of the precarbonization phase of thermostabilized PAN fibres for production of performant carbon fibres. This way, for the precursors we used, the optimal temperature is 800°C and the optimal stationing time is 148...262 seconds.

2. The results obtained by a fuzzy language model relating to optimal temperatures and retention times are according to those obtained by experimental methods.

3. Another advantage of fuzzy language systems utilization is the simplification of optimal conditions establishment, replacing the experimental methods with methods of computing which are more simpler and less expensive.

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*Institute of Research for Chemical Fibres, Săvinești
and
Technical University "Gh. Asachi", Jassy,
Department of Physics*

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MODEL FUZZY PRIVIND OPTIMIZAREA PROCESULUI DE PRECARBONIZARE A FIBRELOR PAN TERMOSTABILIZATE

(Rezumat)

În cadrul procesului de obținere a fibrelor carbon, faza de precarbonizare, care constă în tratarea în mediu inert a fibrelor termostabile la temperaturi în domeniul 350...2000°C, este importantă prin tranzițiile structurale care se produc în fibră. Datorită proceselor complexe, de multe ori simultane, este greu de stabilit un model determinist care să poată conduce procesul. Din acest motiv s-a utilizat un model lingvistic de tip fuzzy.

THE PHOTOELECTRICAL INTERFACE PROCESSES IN MULTILAYER-TYPE HETEROSTRUCTURES BASED ON II-VI SEMICONDUCTORS COMPOUNDS AND PHOTOSYNTHETIC PIGMENT

BY

MAGDA GHERGHEL

Abstract. The studies dealing with the photosynthetic pigments, known as chlorophylls (Chl), proved that these organic compounds possess two important properties necessary for a material used in the preparation of an efficient photovoltaic device, namely a strong absorption in the visible range and an activation semiconduction energy of 1...2 eV, where the calculated efficiencies of the photovoltaic devices are maximum. The considerable photoconductor and photovoltaic properties of the microcrystalline chlorophyll thin films (Chl) are mentioned in the literature. On taking into account these above mentioned considerations, we study the influence which a Metal/pChl/Metal-type Schottky barrier structure could have on some tandem-type heterojunctions based on II-VI, III-V semiconductor compounds, with a view to improve their photovoltaic performances. One presents the results of the experimental researches on the study of the photoelectric and photovoltaic behaviour of a tandem-type mixed multilayer heterostructure (a tandem cell system) of the following form: $Al/n^+nSi-pChl-Cu-nCdS-pCu_xS/Cu$. The suggested photovoltaic system is made up of a Schottky barrier semiconductor structure of the $Al/n^+nSi-pChl/Cu$ type found in interaction with a $nCdS-pCu_xS$ type heterostructure and in series with it. The photoelectric characteristics of the Schottky barrier made evident in a series of works in this domain (for a Metal/pChl/Metal-type structure) reveals the important role which could be played by the introduction in the Metal/pChl interface region of a n^+n -type guard semiconductor structure (with an adjacent surface field) with a view to accomplish the increase of the ϕ_B height of Schottky barrier at the Metal/pChl interface. We have chosen an Al/n^+n polycrystalline Si-type structure (with work function $\phi_{Al} < \phi_{Si}$), which behaving as on non-rectifier contact has the role of inducing the formation of a layer of negative spatial charge accumulation at the Al/pChl interface, thus determining an increase of the interface energetic barrier height.

Key words: tandem cell system, photovoltaic properties, Schottky barrier, quantum tunnelling effect, optical absorption mechanism, negative photoconduction phenomenon, guard semiconductor structure.

1. Introduction. One of the principal problems put in the case of the photovoltaic devices constituted from thin semiconductor layers, is that of the relation between price and productivity, price and technology, price and profitability. On

taking into account the overall magnitude of the conversion output at this type of devices, a special attention has also been given lately to those researches directed towards the accomplishment of some photovoltaic structures with Schottky barrier (BS) and of some tandem heterostructures photovoltaic systems. Being constituted from more series-superposed and vertically-illuminated semiconductor junctions, the tandem-type photovoltaic systems allow the selective use of a part as large as possible from the spectrum of the solar radiations which is of interest for photovoltaic conversion. On the other hand, the increase of the conversion efficiency is strongly connected with the problem of reducing the cost price, a problem which can also be workable by using some new semiconductor materials as cheap and as easy to obtain as possible.

From this category of new materials, there belong as well the photosynthetic pigments — the microcrystalline chlorophyll (Chl) — which present a strong absorption in the visible range and an activating energy for photoconduction of 1...2 eV (for which the calculated efficiencies of the photovoltaic devices are maximum) [6], [9]...[11]. Also, the height of the energy barrier ϕ_B constitutes an important electrical parameter of a Schottky barrier heterostructure and that is why in order to increase the photovoltaic conversion efficiency, the Schottky structures need barriers ϕ_B as high as possible [1]...[6], [9]...[11]. This happens because the height ϕ_B of a Schottky barrier is the parameter which strongly influences the value both of the reverse current and of the photovoltaic voltage in an open circuit U_{CD} at the multijunction-type heterostructures with a Schottky barrier. Consequently, it is necessary to exist the possibility of changing the height ϕ_B of the barrier between limits as large as possible, the checking of the barrier height being done as well by the changing of the doping of the semiconductor in a superficial layer at the Metal/Semiconductor interface [1], [3], [4], [6].

On taking into account the grounds presented above, and also considering the fact that the photosynthetic pigments show an intense absorption in the visible range, as well as the fact that the microcrystalline chlorophyll films show considerable semiconduction properties [6], [9]...[11], we have proposed to study the influence of an Al-Chl-Cu type heterostructure on the CdS-Cu_xS junction, with the aim of using the photosynthetic pigment thin films in producing mixed photovoltaic devices based on anorganic semiconductor materials.

The photoelectrical characteristics of the Schottky barrier, made evident at the Al/pChl interface [6], [9]...[11], have determined us to consider the important role which could be played by the introduction — in the interface region — of a guard semiconductor structure [1]...[5] (with an adjacent superficial field) of the n^+ type with a view to accomplish an increase in the height ϕ_B of the interface energy barrier. That is why, we have made our choice to select the Al/ n^+ nSi-Chl-Cu-CdS-Cu_xS/Cu, made up of a Schottky barrier semiconductor structure of the Al/ n^+ nSi-pChl/Cu type, found in interaction with a nCdS-pCu_xS type heterostructure and in series with it.

2. Experimental Results. Discussions. The CdS-Cu_xS heterojunction has been produced by vacuum evaporation, sputtering and dipping methods [1], [2], [4], [9]...

[11] on the $\text{Al}/n^+\text{Polycrystalline Si}$ sublayers area $2.5 \times 3 \text{ cm}^2$, $125 \dots 150 \text{ }\mu\text{m}$ thickness and with an electrical resistivity ranging between $5 \dots 10 \text{ }\Omega\text{m}$. The Al films had a $0.5 \dots 1.5 \text{ }\mu\text{m}$ thickness while the thickness of the $n^+\text{Si}$ diffused layer was between $0.2 \dots 0.3 \text{ }\mu\text{m}$.

On taking into account some advantages of the Al/polycrystalline Si system, namely, the simplicity and the low cost of the technology, the good adherence of Al on Si, the Al quick selfpassivation by superficial oxidation [1]...[5], we have chosen the Al/ n polycrystalline Silicon structure as a suport for the $\text{CdS-Cu}_x\text{S}$ heterojunction. Another important argument in choosing the $\text{Al}/n^+\text{Si}$ structure is the fact that a strong doping — of the n^+ type of the polycrystalline Silicon — with donor-type impurities (As, $N_d = 5 \dots 20 \cdot 10^{21} \text{ cm}^{-3}$), can determine an increase of the Si atomic radius by negative ionizing ($R_{\text{Si}} = 2.71 \text{ \AA}$), thus leading to the solwing down of the diffusion of the Si atoms in Al, the n -type impurities accomplishing, at the same time, a saturation of some unsatisfied links at the Si intergranular boundaries, they being passivated as compared with the Al interaction [3]...[5]. Simultaneously, in the presence of the superficial field adjacent to the $\text{Al}/n\text{Si}$ contact, by means of the action of the dopped n^+ diffused layer, the minority photocarriers will be turned back and directed towards the separating field of the barrier region, thus contributing to the increase of the short-circuit current density ($|J_{\text{SC}}| = |J_L|$) by increasing the collecting efficiency. Accordingly, on using as a sublayer the $\text{Al}/n^+\text{Si}$ (non-rectifier contact) and taking into account the fact that the thin films of microcrystalline Chl present remarkable photoelectrical properties characteristic for the p -type photoconduction [6], [9]...[11], a Schottky barrier heterostructure of the $\text{Al}/n\text{Si-pChl-Cu}$ form has been accomplished.

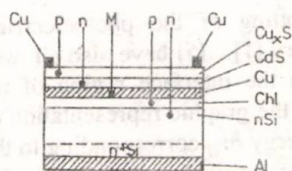


Fig. 1 — The $\text{Al}/n\text{Si-Chl-Cu-CdS-Cu}_x\text{S/Cu}$ heterostructure.

made up of two series connected photojunctions ($\text{Al}/n^+\text{Si-pChl}$ and $n\text{CdS-Cu}_x\text{S}$ respectively) and having the form $\text{Al}/n^+\text{Si-pChl-Cu-nCdS-pCu}_x\text{S/Cu}$ (Fig. 1).

The CdS and Cu_xS component layers had thickness ranging between the $20 \dots 25 \text{ }\mu\text{m}$ and $0.15 \dots 0.20 \text{ }\mu\text{m}$ intervals, respectively, while the thickness of the Cu films was between $0.1 \dots 0.5 \text{ }\mu\text{m}$.

We mention the fact that $\text{Al}/n\text{Si-Chl-Cu-CdS-Cu}_x\text{S/Cu}$ mixed tandem-type photoelement which finally resulted, being made up of more semiconductors characterized by various values of the forbidden band ($E_{g\text{Si}} \approx 1.1 \text{ eV}$, $E_{g\text{Chl}} \approx 1.7 \text{ eV}$, $E_{g\text{CdS}} \approx 2.4 \text{ eV}$, $E_{g\text{Cu}_x\text{S}} \approx 1.2 \text{ eV}$) has the possibility of using a great part of the solar radiation spectrum, allowing, at the same time, a bifacial illumination which improves

The thin microcrystalline Chl films have been prepared by sputtering, electrodeposition and dipping methods [9]...[11], these films having a $0.5 \dots 5 \text{ }\mu\text{m}$ thickness and an average value of resistivity of $3 \cdot 10^{11} \text{ }\Omega\text{cm}$. The $\text{CdS-Cu}_x\text{S}$ heterojunction has been superposed on the $\text{Al}/n\text{Si-pChl-Cu}$ structure by alternative deposition of the component photoconductor films [1]...[6], [9]...[11], finally results a mixed multi-junction-type heterostructure (tandem cell system),

the increase of the collecting efficiency.

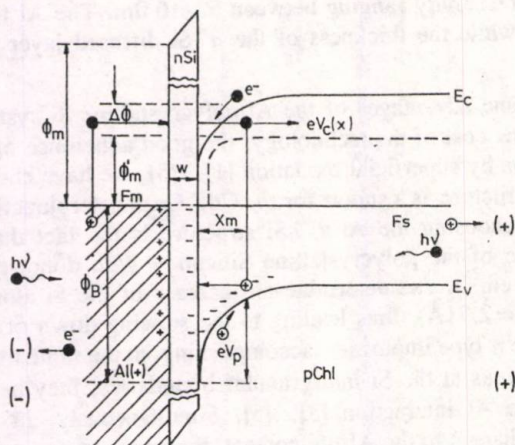


Fig. 2 — The barrier layer at the Al/p-Chl interface.

Accordingly, at the Al/Chl interface, a barrier layer would be formed, one containing a negative spatial charge (electrons), a charge due both to the ionized acceptors remained uncompensated by the disappearance of the holes in the pigment film on a distance x , and to the layer of negative spatial charge accumulation on a distance w from the non-rectifier Al/ n^+n Si contact (Fig. 2).

The photoelectrical measurements for the plotting of the photoelectrical characteristic $J_L = f(V)$ by applying Crowell's theory [1]...[5] have also allowed us to determine the height of the Schottky barrier in the interface region of the Al/ n Si- p Chl structure, bifacial illuminated. Thus, from the graphic representation of the relation $J_L = (h\nu - h\nu_0)^2 = (h\nu - \phi_B)^2$, the photonic energy $h\nu_0$ corresponding to the height of the Schottky barrier, $h\nu_0 = eV_B = \phi_B$, there resulted a straight line for values of the energy of the photons flux, $h\nu > \phi_B$.

By extrapolating this straight line up to the intersection with the energy axis $h\nu$, in the intersection point there have been obtained average values for the height of the barrier ϕ_B , values ranging in the interval $\phi_B = 0.95 \dots 1.20$ eV (Fig. 3).

The photoelectrical changes experimentally found out at the Al/ n^+n Si- p Chl-Cu structure have led us to conclude that in the study of the interdependences between the photovoltaic parameters and the height ϕ_B of the Schottky barrier, at the interface region, it is necessary to take into account both the tunnelling and the recombination effects of the minority carriers and the influence of the image forces (in the spatial charge region) on the potential barrier V_B corresponding to the Schottky barrier [1], [3], [4], [9]...[11].

Therefore, when estimating the diminishing $\Delta\phi_B$ of the Schottky barrier height

In some previous papers [9]...[11], we had opportunity to show that, during our researches on the photoelectrical behaviour of the Al-Chl-Cu heterostructure, we noticed both the fact that instead of the saturation phenomenon of the reverse circuit there appeared a slight variation of it with the external voltage, i.e., a dependence of the form $J_S = f(V^{1/2})$ and a p -type semiconduction in the thin layer of the photosynthetic pigment; finally, a Schottky barrier was rendered evident at the Al/ p Chl interface ($\phi_{Al} < \phi_{Chl}$).

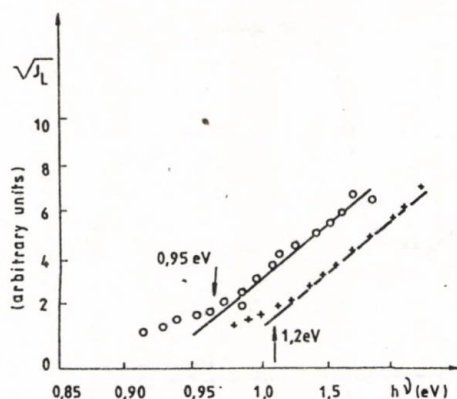


Fig. 3 — The measured photoelectrical characteristics.

a quantum tunnelling effect — thermically non-activated, of the electronic flux through the interface states, in the presence of the superficial electric field adjacent Al/nSi contact (field emission). The effect of the Schottky barrier diminishing $\Delta\phi_B$ is the increase of the photocurrent tunnelling component. At the same time, we assume that it appears the phenomenon of returning into the metal of the electrons which climbed over the barrier as a result of their interaction with the phonons [9]...[11] in the region of the higher wavelength, $h\nu \approx 0.90$ eV (Fig. 3), where the frequency of the photonic radiation is comparable with the frequency of the dipole oscillation from the structural lattice of the thin pChl films.

During the experiments and for the samples, it has been noticed that after their continuous illumination for a time interval of approximately $t \approx 2$ minutes, the negative photoconduction phenomenon undergoes a saturation process, passing in a positive photoconduction regime with the establishing of the photovoltaic parameters (Fig. 4).

The existence of the negative photoconduction phenomenon at room temperature, rendered evident in this type of heterostructure with Schottky barrier, have suggested us a close interaction and interdependence between the diminishing $\Delta\phi_B$ of the barrier (in the Al/n⁺Si-Chl interface region), the structure of the intercrystalline boundaries at the Al/n⁺Si interface, the recombination processes of the minority photocarriers along the interface intergranular boundaries, the gettering processes of the Cu⁺ and Mg⁺⁺ impurities existing at the intercrystalline boundaries of the interface region (Al/n⁺Si-Chl and Chl/Cu, respectively). Thus, we consider that the appearance of the negative photoconduction phenomenon for this type of structure could be partly due to the existence of some acceptor levels of Mg⁺⁺ existing at the intercrystalline interface boundaries, levels which can constitute traps for some optically active centers for electrons in the n⁺Si/pChl interface region (Fig. 5).

Accordingly, on considering, on one hand, the structure of the intergranular

[1], [5], [9]...[11],

$$\Delta\phi_B = \frac{e^3 E}{4\pi\epsilon_0\epsilon_s}, \quad \epsilon_s = 12...16\epsilon_0,$$

$$\Delta\phi_B = 0.020 \dots 0.040 \text{ eV},$$

we have taken into account the presence of the superficial electric field adjacent to the contact Al/nSi ($E \approx 10^4 \dots 10^5$ V/cm) as well as the correction introduced by the image forces in the Al/nSi-pChl interface region. On considering the mixed structure of this multijunction as well as the excitation state of the Mg⁺⁺ atoms included in the pigment molecule [9]...[11], we ponder the existence of

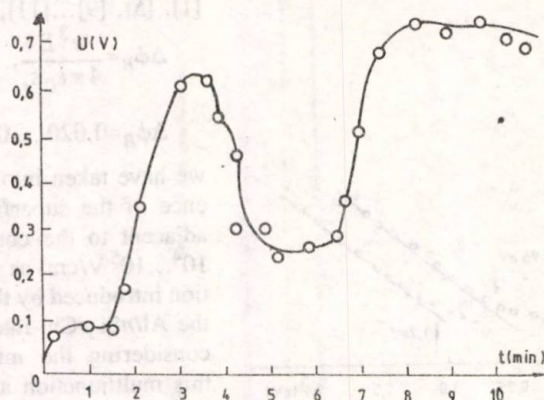


Fig. 4 — Variation of the photovoltage at room temperature, with illumination time.

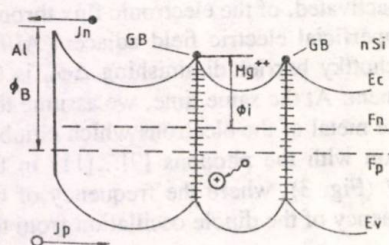


Fig. 5 — Interface states in the region of the intercrystalline boundaries at the Al/ n^+ Si-Chl structure.

minoritary photocarriers on the states from the intercrystallites boundaries, a fact which explains the value decrease of the photocurrents and photopotentials and, thus, the appearance of a negative photoconduction regime (Fig. 5). We also consider that the negative photoconduction phenomenon at this type of structures is linked with a mechanism of decreasing the photocarriers mobility under the influence of the adjacent superficial electric field existing at the Al/ n Si interface.

When accomplishing the photoelectrical measurements for the Al/ n Si-Chl-Cu-Cds-Cu_xS/Cu multijunction-type heterostructure, it has been noticed that for average values of the illumination time $t \approx 4$ minutes, the amplitude of the negative photoconduction is saturated, passing then in a positive photoconduction regime with the establishing of the photovoltaic parameters.

3. Conclusions. 1. The study of the photoelectrical behaviour of the mixed

boundaries at the Al/ n^+ polycrystalline Si — rendered evident by the SEM electronical micrographs techniques, TEM [3]...[8], and on the other hand, the excitation state of the Mg^{++} atoms included in the $pChl$ photosynthetic pigment molecule [9]...[11] as well as the possible diffusion of these atoms along the intercrystalline boundaries existing in the interface region ($R_{Mg^{++}} \approx 0.65 \text{ \AA}$) we believe that in the illumination time interval corresponding to the negative photoconduction, $t \approx 3 \dots 6$ minutes, there takes place a strengthening of a quick recombination process of the

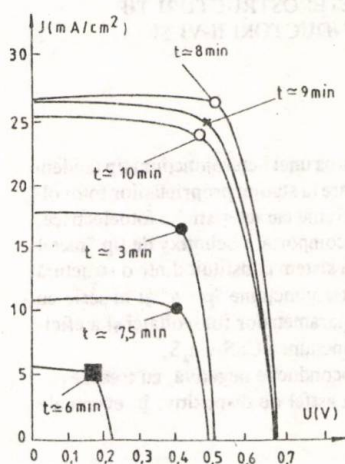


Fig. 6 — The J-V characteristic for the $\text{Al}/n^+\text{Si}-\text{Chl}-\text{Cu}-\text{CdS}-\text{Cu}_x\text{S}/\text{Cu}$ structure.

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Technical University "Gh. Asachi", Jassy,
Department of Physics

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heterostructure with Schottky barrier suggested by us as having the form $\text{Al}/n^+\text{Si}-\text{Chl}-\text{Cu}-\text{CdS}-\text{Cu}_x\text{S}/\text{Cu}$ has emphasized the fact that the thin films of photosynthetic pigments (whose obtaining is simple and implies very low cost price) can determine a substantial improvement of the photoelectrical parameters characteristic to the photovoltaic systems.

2. For this mixed tandem-type heterostructure the following average values for the characteristic photoelectrical parameters have been obtained (under laboratory conditions): $J_{SC} \approx 25 \text{ mA/cm}^2$, $U_{CD} \approx 675 \text{ mV}$ (Fig. 6), collecting factor $C_F \approx 0.75$, photovoltaic conversion efficiency $\eta \approx 15 \dots 18 \%$, the form of the current-voltage characteristics for this type of photoelement was in agreement with the experimental results we obtained by photoelectrical measurements.

PROCESE FOTOVOLTAICE DE INTERFAȚĂ LA HETEROSTRUCTURI TIP
MULTISTRAT PE BAZĂ DE COMPUȘI SEMICONDUCTORI II-VI ȘI
PIGMENTI FOTOSINTETICI

(Rezumat)

Sunt prezentate atât rezultatele experimentale privind realizarea unei heterojoncțiuni tip tandem de forma $Al/n^+nSi-Chl-Cu-CdS-Cu_xS/Cu$, cât și concluziile privitoare la studiul proprietăților fotovoltaice și electrooptice ale acestui sistem. Cercetările experimentale privind caracteristicile fotoelectrice, au scos în evidență faptul că structura $Al/n^+nSi-Chl/Cu$ prezintă o comportare Schottky de tip "metal / p semiconductor", întregul dispozitiv fotovoltaic funcționând ca un sistem constituit dintr-o structură semiconductoare cu barieră Schottky, aflată în interacțiune cu o heterojoncțiune " $p - n$ " și în serie cu aceasta, ceea ce a condus la o îmbunătățire substanțială a valorilor parametrilor fotovoltaici și a eficienței de conversie la acest tip de structură, comparativ cu heterojoncțiunea $CdS-Cu_xS$.

Totodată, s-a pus în evidență existența unui fenomen de fotoconducție negativă, cu trecere periodică în fotoconducție pozitivă, ceea ce sugerează utilizarea unui astfel de dispozitiv, în eventuale aplicații optoelectronice.

ON THE STRUCTURE INDUCED TRANSFORMATIONS BY $^{32}\text{S}(+5)$ IMPLANTATION AT SOME Fe-Cr-Ni ALLOYS

BY

M. L. CRAUS and EMIL LUCA

Abstract. The austenitic Fe-Cr-Ni alloys behave some particularities concerning the phase composition and the residual microstrain. The structure modification due to thermomechanical annealing and irradiation with $^{32}\text{S}(+5)$ ions are studied on Incolloy 800 type alloy.

Key words: austenitic alloy, phase composition, residual microstrain, structure modification, thermomechanical annealing.

1. Introduction. The Fe-Cr-Ni austenitic alloys are characterized by the stability of mechanical properties at high temperatures, resistance against acids and oxidation. Such alloys are used as structure materials in CANDU 600 reactors [1]...[3].

The irradiation could induce metallic atoms shiftings from their position in crystalline lattice and the increase of interstitial atom and vacancy concentrations. As a secondary effect the irradiated voids appear and the embrittlement of the material, a modification of the creep behaviour, the precipitation of carbides may occur. The first stage of irradiation is associated with the dislocation loops. These defects can appear after the thermomechanical treatments.

The annealings between 550 and 700°C and the irradiation with neutrons or heavy ions could induce the $\gamma\text{-Ni}_3(\text{Al}, \text{Ti})$ segregation (in Ni-rich alloys like Inconel 600) [4] or the appearance of a martensitic phase [5].

The defects (dislocations, precipitate particles, voids etc) behave like the traps for hydrogen or neutrons and can contribute to the mechanical strength decrease in the alloys.

Our aim is to study the structural transformations due to the irradiation of some Fe-Cr-Ni alloys with $^{32}\text{S}(+5)$ ions.

2. Experimental Methods. The alloy Incolloy 800 was obtained by melting of the electrolytic nickel, pure iron, metallic chrome, iron-titan alloy, iron-silicium alloy

mixture. The chemical composition of this alloy was checked by chemical and spectral analysis (see Table 1).

Table 1

Sample	Fe	Ni	Cr	Si	Mn	Al	Ti	Cu	C	S
A1	43.2	33.0	21.0	0.6	0.8	0.5	0.5	0.4	0.05	0.01

The spectral analysis was performed with the VRA 20R X-ray spectrometer. The analyses of the phase composition, the lattice constants, the microstrains and the crystallite size were made with a X-ray diffractometer DRON-2,0, which has a vacuum high temperature enclosure. The lattice constants determined at room temperature were corrected by using pure nickel as external standard. The centroid position of the diffraction maximum was determined by means of a method proposed by Craus and coworkers [7]. The crystallite size was found by means of the metallographic analysis. If the average size of the crystallites is great enough (greater than 2 μm), the halfwidth of the diffraction maximum, corrected for instrumental errors by a method proposed by Craus and coworkers [8], follows the dependence [9]

$$(1) \quad \frac{\beta_2}{\beta_1} = \frac{\tan \theta_2}{\tan \theta_1}.$$

This method allows the separation of the strains and the size contributions [7].

The metallographic analysis of the sample was made with a Zeiss-Jena microscope; it allows to determine the average size of the crystallites and to detect the foreign phases.

Table 2

Thermomechanical treatment code	Thermomechanical treatment	Obs.
A1 ₁	As casted — 770 — 1120 — 1300°C	nonirradiated
A1 ₂	As casted — 1020 — 758 — 885 — 1070°C	nonirradiated
A1 ₃	As casted — 450 — 550 — 670 — 800°C	irradiated

Some samples were irradiated within the TANDEM accelerator (IFIN-București), on a 4...5 mm² area, with pentavalent ions ³²S. The total number of implanted ions was about 3.6 · 10¹⁴ (E=42 MeV, 10¹⁶...10¹⁷ nuclei/cm²). The penetration depth, calculated with a TRIM routine, is about 5 μm , and the spreading about 0.4 μm .

The samples were annealed at high temperature between 400 and 1300°C, in vacuum (Table 2). The irradiated samples were annealed between 450 and 800°C, in vacuum. The X-ray and metallographic analyses of the nonannealed samples do not

indicate the presence of any foreign phase.

3. Results and Discussions. From the high temperature diffractograms, one could conclude:

- i) the temperature dependence of the lattice constant is not linear;
- ii) there is a dependence of the lattice constant variation vs. initial thermomechanical treatment.

The samples which were initially treated at lower temperatures (770°C), have a much slower dependence of the lattice constants vs. temperature, than those aged at 1020°C (Fig. 1).

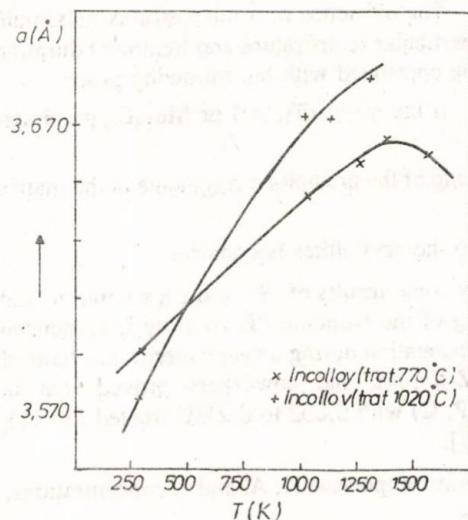


Fig. 1 — The lattice constant vs. treatment temperature.

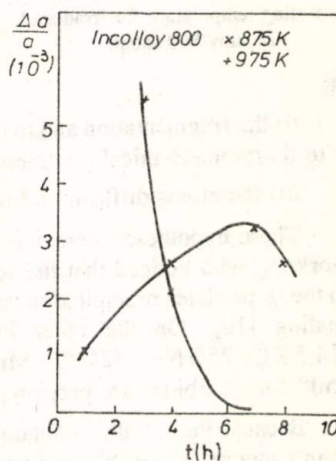


Fig. 2 — The dependence of the strain on temperature and the treatment duration.

The samples initially annealed at 770°C are strongly textured a great number of the crystallites being oriented so that their (200) planes are parallel with the samples plane. This orientation is very marked at 1300°C. The samples, which were annealed at 1020, 758, 885, 1070 and 700°C and measured at 20°C, have a normal distribution of the maximum intensities, with a slight rise of the preferential orientation of the crystallites vs. treatment period.

The sample annealed and measured at 1020°C has a preferential orientation of the crystallites (a great number of crystallites have (200) planes parallel with the sample plane), but this orientation disappears at the next treatment. The measurements performed at 600°C indicate a maximum of the lattice strains after 7 h of treatment, while for a 700°C treatment we can observe a fast decrease of the lattice strains (Fig. 2).

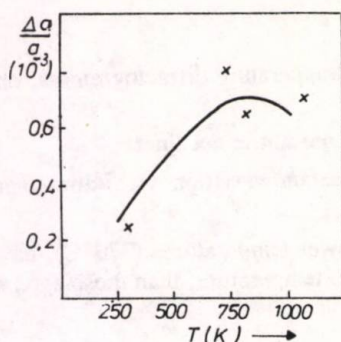


Fig. 3 — The microstrains variation vs. annealing temperature for irradiated Incolloy 800 sample.

The Incolloy 800 samples irradiated with $^{32}\text{S}(+5)$, annealed in vacuum between 20 and 800°C has a microstrains maximum at 500°C.

The maximum value is four times smaller as those corresponding to the nonirradiated samples, at 600°C (Fig. 3).

The X-ray analysis pointed out a small amount of the foreign phase in the irradiated Incolloy 800 sample. This phase is present in the sample until 800°C.

The existence of a microstrains maximum at a particular temperature and treatment duration can be correlated with the following processes:

- i) the γ' - $\text{Ni}_3(\text{Ti, Al})$ or Me_{23}C_6 precipitation;
- ii) the fragmentation and the resolving of the precipitate fragments in the matrix due to thermomechanical treatment;
- iii) the atoms diffusion (chrome) to the crystallites boundaries.

These hypotheses were suggested by some results of Sundaraman and coworkers, who noticed that the softening of the Nimonic PE-16 alloy is associated with the γ particles precipitation and fragmentation during a cyclic thermomechanical annealing [10]. On the other hand, Zhang and coworkers proved that in Fe-14.5%Cr-25%Ni-0.32%(Si, Mn, W, P, C) with 0.002 to 0.2%C treated for 24 h at 850°C the carbides are precipitated [11].

Because the Al alloy contains a relative important Ti, Al and C concentrations, we can suppose as possible two processes:

- 1) the Cr_{23}C_6 carbide appearance;
- 2) the γ' - $\text{Ni}_3(\text{Al, Ti})$ phase precipitation.

The Cr_{23}C_6 concentration decrease does not influence the lattice constant and the strains, while the diminution of C(0.77 Å), Ti(1.47 Å) or Al(1.43 Å) concentrations could contribute to the lattice constant and strains decreasing.

The appearance into the crystallites of precipitated particles, which have a different lattice as compared with those of the matrix, could generate strains into the lattice matrix. The segregation of carbon atoms, which occupied interstitial places, as carbides and the precipitation of γ' phase produce strains with different sign. The samples, which were annealed at 600 and 700°C, have different lattice constants and microstrains: 3,819 Å and $0.02 \cdot 10^{-3}$ and 3,5802 Å and $1.75 \cdot 10^{-3}$, respectively. The difference between the lattice constants is due to the different Al, Ti, C concentrations into the matrix. Concerning the irradiated Incolloy 800 type alloy samples, the strains

maximum seems to be due to a foreign phase precipitation, which was observed in X-ray diffractograms, and to a recrystallisation of the matrix. The annealing diminishes the microstrains of the γ phase, while the γ' phase precipitation leads to an increase of the microstrains, because the unit cell volume of the γ -phase is greater than those of phase γ' ($V(\gamma)=42.42 \text{ \AA}^3$; $V(\gamma')=45.38 \text{ \AA}^3$).

The irradiation effects are:

- i) a general diminution of the microstresses from the irradiated part of the sample;
- ii) the appearance of a small foreign phase quantity.

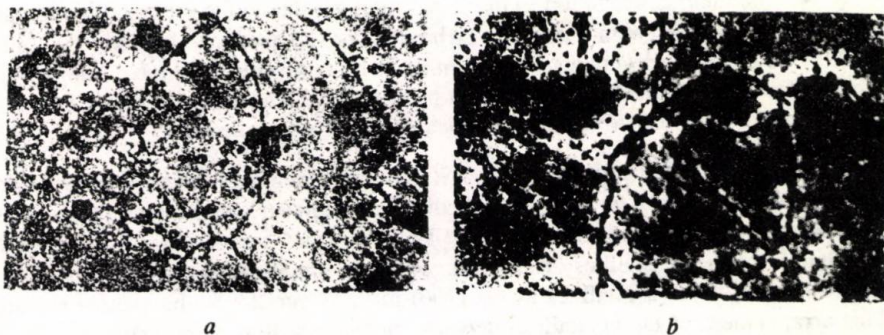


Fig. 4 - a) the irradiated sample ($T_{\text{treatment}}=800^\circ\text{C}$); b) the nonirradiated sample ($T_{\text{treatment}}=600^\circ\text{C}$).

The atoms positions could be modified as the result of their knockings with $^{32}\text{S}(+5)$ ions. Because the activation energy of the displacement process is about 30 eV, one $^{32}\text{S}(+5)$ ion could produce several thousand atomic displacements. An important part of the displaced atoms are setting on the positions, so as diminish the local energy of the lattice and the microstrains, too. This process takes place together with the migration of some atoms (S, C, Al, Ti), which have an atomic radius different from that of the place they occupied in the lattice, to the crystallite boundary and the appearance of the M_{23}C_6 carbide or/and γ' -Ni₃(Al, Ti) phase [12].

The metallographic analysis of the Incolloy 800 irradiated and annealed at 800°C in vacuum (Fig. 4 a) indicates the presence of some foreign phase crystallites and a fragmentation of grains in the irradiated part of the sample, confirming the studies made by X-ray methods.

4. Conclusions. 1. The microstrains of Incolloy 800 samples can increase defaults concentrations (fine particle precipitating within the matrix or at the crystallites boundaries, dislocations, Frenkel defaults). The decrease of average size of the crystallites with the duration and the temperature of the treatment depends upon the chemical composition of the alloy and is small for Incolloy 800 [12].

2. The solution heat treatment and the chemical composition of the nonirradiated samples influence the temperature corresponding to the maximum values of the microstrains. The strains variation are due to: the Cr_{23}C_6 or/and $\gamma\text{'-Ni}_3(\text{Al, Ti})$ precipitation; the fragmentation of the crystallites.

3. The diminution of the average size of the crystallites increases the surface energy, but decreases the stresses energy. The strain are diminished several times due to irradiation. The variation of these microstrains vs. annealing temperature depends on the chemical composition of the alloys: the Inconel 600 type alloy has a microstrain minimum at about 550°C [12], while the Incolloy 800 type alloy has a microstrain maximum at about 470°C .

4. The main contribution to these variation are due to the recrystallization and the precipitation of foreign phase like carbides (Cr_{23}C_6) and/or $\gamma\text{'-Ni}_3(\text{Al, Ti})$. In the Incolloy 800 irradiated samples, the maximum is probably due to a higher temperature recrystallization, where the precipitation processes ceases. It is also possible that at higher temperature the precipitate particles are resolved into matrix.

5. A short time irradiation with $^{32}\text{S}(+5)$ ions diminishes the microstresses into the irradiated samples. The annealings modify the irradiated samples texture much less than that of the nonirradiated samples. The irradiated samples have an isotropic distribution of the crystallites orientation, while the nonirradiated samples have such an orientation of the crystallites as the (200) plane is parallel to the sample surface. This arrangement of the crystallites depends on the annealing temperature.

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Institute for Technical Physics, Jassy
and
Technical University "Gh. Asachi", Jassy,
Department of Physics

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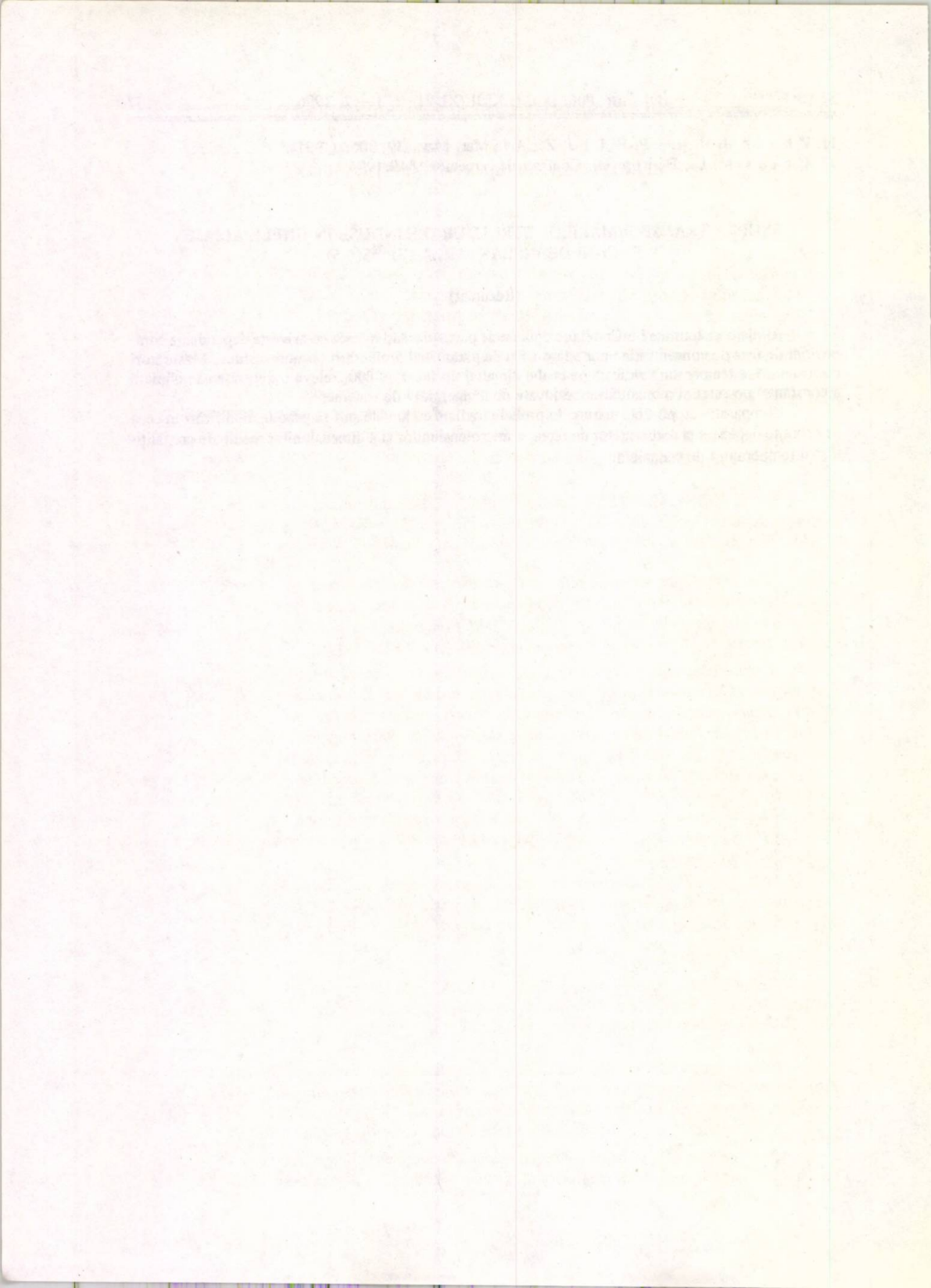
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ASUPRA TRANSFORMĂRIILOR STRUCTURALE INDUSE ÎN UNELE ALIAJE
Fe-Cr-Ni DE INPLANTAREA CU $^{32}\text{S}(+5)$

(Rezumat)

Aliajele austenitice Fe-Cr-Ni prezintă unele particularități în ceea ce privește dependența compoziției de fază de concentrația unor adaosuri și de parametrii prelucrării termomecanice. Măsurători de structură la temperatură ridicată pe probe din aliaj tip Incolloy 800, relevă o dependență neliniară a constantei de rețea și a tensiunilor reziduale de temperatura de tratament.

Comparativ cu probele martor, în probele iradiate cu ioni de sulf se produc modificări în ceea ce privește dependența constantelor de rețea, a microtensiunilor și a dimensiunilor medii ale cristalitelor cu temperatura de tratament.



MAGNETIC PROPERTIES OF THE Mn-Zn FERRITES IMPURIFIED WITH TiO_2

BY

SERGIU ISTRATE

Abstract. The characteristic quantities of the hysteresis loop at room temperature are obtained and analyzed, as well as the graphs of the initial permeability function of temperature for a Mn-Zn ferrite impurified with TiO_2 .

Key words: Mn-Zn ferrite, hysteresis loop, initial permeability.

1. Introduction. Manganese-zinc ferrites, because of their high permeabilities and low losses at frequencies ranging up to several hundred kilocycles per second, have important applications in telecommunications industry. The highest permeability are usually obtained from starting composition containing a small excess of iron oxide. It is well known that additives in ferrites play an important role in improving their magnetic characteristics. For instance the role of small amount of Ca and Si in the high permeability of the Mn-Zn ferrites was investigated.

The addition of the relative small amount of TiO_2 in Mn-Zn ferrites produces important modifications of the magnetic and electric properties [1], [2]. With a view to improve the performances of the high permeability Mn-Zn ferrites, with applications in telecommunications, we studied the variation of the initial permeability and the modifications of the hysteresis loop characteristics, respectively, versus the additions of small amounts of TiO_2 and different sintering temperatures.

2. Experimental Procedure. 2.1. Preparation Method. The raw materials — oxides — had a purity of 99.6%. We chose the composition 25.5 mol% MnO, 22 mol% ZnO, and 52.5 mol% Fe_2O_3 who presents a high permeability at room temperature, and another three compositions with the addition of 0.66, 1.32, and 1.98 g% TiO_2 , respectively beyond the stoichiometry. The powders was homogenized, presintered, milled and pressed in the same conditions. We utilize samples of toroidal shape and with the same dimensions. The samples were sintered at the temperatures 1220, 1250, 1280, and 1350°C respectively, during 5 hours. The cooling was slow in the furnace in N_2 atmosphere.

2.2. Measuring Equipment. We determined the variation of the initial permeability versus temperature in the range -150°C and Curie temperature, the first magnetization curve and the hysteresis loop, at room temperature, from which we deduced B_s , B_r , H_c and the ratio $\mu_{\max}/\mu_i$.

The initial permeability was measured on the toroidal probes, provided for the winding, and computed with the formula

$$\mu = \frac{0.217 \cdot 10^7}{n^2 h \lg(d_e/d_i)} L, \quad (1)$$

where d_e is the external diameter of the toroid, d_i — internal diameter, h — thickness in mm, n — number of turns and L — the inductance in mH.

The measuring equipment for the inductance was a Maxwell bridge. The probe's winding was in series with an ohmic resistance. A selective millivoltmeter measures the voltage drop on the ohmic resistance and in this way we determine the current in the winding and then the magnetic field in the samples. The measurements were performed in a magnetic field of 5 mOe at frequency 1 kHz. The sample was placed in a reglable thermostat.

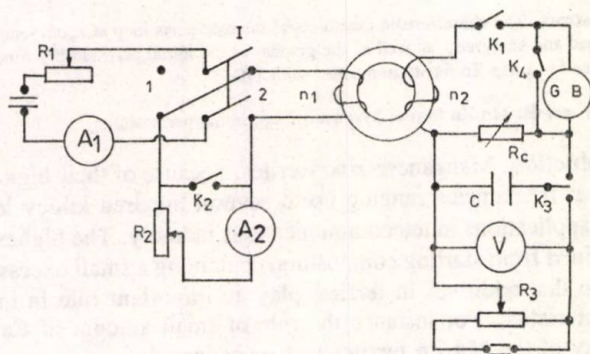


Fig. 1 — Measuring equipment for hysteresis loop.

The schema of the installation for the hysteresis loop is presented in Fig. 1. The intensity of the magnetic field is $H = n_1 I / l$, where n_1 is the primary number turns, I — current intensity, and l — average length of the magnetical circuit; H results in A/m.

Magnetical induction is obtained [3] with the relation

$$B = \frac{C E r}{S n_2 \alpha_0'} \alpha = k \alpha, \quad (2)$$

where C — is the capacity for gauging in F, E — voltage for gauging in V, r — resistance of galvanometer circuit in Ω , S — the area of the sample's section in m^2 , n_2 — the secondary number turns, α_0' — the galvanometer deviation at the condenser discharge, α — the galvanometer deviation from a measuring; B is expressed in T.

3. Results and Discussions. In the Fig. 2 we represented the curves $\mu_i = f(T)$ determined on the samples sintered at 1250°C . We see that secondary maximum are

not well-marked. However exists the domains in which μ_i varies just slow versus temperature, between an inferior temperature T_k , where the curve $\mu_i=f(T)$ diminishes her slope and a temperature about 20 °C lower than Curie temperature.

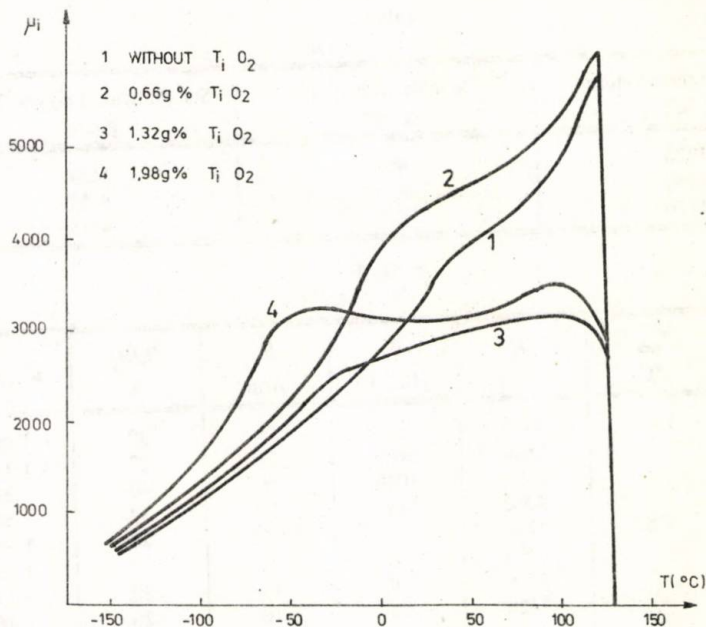


Fig. 2 — Initial permeability versus temperature.

The temperature T_k decreases with the content of titanium from about +20°C for the ferrite with 0.66 g % TiO_2 to about -50°C for the ferrite with 1.98 g % TiO_2 . The shifting of the temperature is roughly the same magnitude order with the displacement of the secondary maximum reported in [2]. In contrast with the results determined by Röss and Hanne [2], when the initial permeability from bearing diminishes with the addition of TiO_2 , from Fig. 2 we conclude that at a small addition (0.66 g % TiO_2) the initial permeability increases by comparison with the sample without TiO_2 . This increase of the initial permeability when we added 0.66 g % TiO_2 could be explained by the growth of the density. From Table 1 we observe that the addition of 0.66 g % TiO_2 increases notable the density. Also, according to [2] where the addition of TiO_2 increases Curie point of the ferrite, in our ferrites are not observed a variation from T_c induced by TiO_2 addition.

In the Table 2 we present the characteristics determined by the hysteresis loop, for all studied samples. It is found that these characteristics vary in the same manner versus TiO_2 addition for all sintered treatment, that is every characteristics presents a better value in the case of the samples with 0.66 g % TiO_2 , comparatively with the samples without titanium, and for samples with more TiO_2 ; these values are, as rule,

worse. Then the saturation induction is larger, the magnetic coercitive field is smaller, the initial permeabilities are larger, smaller being the ratios B_r/B_s and $\mu_{\max}/\mu_i$, for the samples with 0.66 g% TiO_2 .

Table 1

Sintering temperature °C	Sample without Ti g/cm ³	Sample with 0.66 g% TiO_2 g/cm ³
1220	4.05	4.24
1250	4.36	4.48
1350	4.46	4.78

Table 2

TiO_2 g%	$T_{\text{sint.}}$ °C	B_s 10 ⁻⁴ T	B_r 10 ⁻⁴ T	H_c A/m	B_r/B_s %	$\mu_{\max}/\mu_i$
-	1120	2460	960	13	39	1.60
0.66		2840	810	11	28	1.14
1.32		2600	1040	16	40	1.34
1.98		3060	1130	18	37	1.50
-	1250	2710	770	10	28	1.28
0.66		3200	700	8	22	1.16
1.32		2860	870	12	30	1.17
1.98		3120	810	10	26	1.36
-	1280	2740	1110	16	41	1.63
0.66		3060	810	9	26	1.20
1.32		2940	1000	9	34	1.58
1.98		3050	1080	8	35	1.33
-	1350	2930	1130	9	39	1.75
1.32		3320	1020	6	31	1.45
1.98		3260	1090	7	33	1.74

The characteristics presented in Table 2 depend also on the sintering temperature. The most favourable treatment is that at 1250°C, which gives initial permeabilities relative large and the ratio $\mu_{\max}/\mu_i$ close to 1, necessary in many practical applications.

4. Conclusion. The small addition (impurity) of TiO_2 in Mn-Zn ferrite improves the magnetic properties. This conclusion is in agreement with [4] where one affirms that impurities of the nemagnetic ions in the ferrites take a turn for improving their characteristics.

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PROPRIETĂȚI MAGNETICE ALE FERITELOR Mn-Zn IMPURIFICATE CU TiO_2

(Rezumat)

Se determină și se analizează mărimile caracteristice ciclului de histerezis, la temperatura camerei, precum și curbele permeabilității inițiale funcție de temperatură în cazul unei ferite de Mn-Zn impurificată cu TiO_2 .

REFERENCES

1. H. H. Henshaw, "The Effect of Temperature on the Growth of the Silkworm, *Bombyx mori* L.", *Ann. Entomol. Soc. Amer.* 1914, 7: 1-10.
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3. H. H. Henshaw, "The Effect of Temperature on the Growth of the Silkworm, *Bombyx mori* L.", *Ann. Entomol. Soc. Amer.* 1914, 7: 1-10.

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RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

KARLHEINZ SPINDLER, **Abstract Algebra with Applications. Vol. I, Vector Spaces and Groups**, Marcel Dekker, Inc., New York, 1994, 756 pp., ISBN 0-8247-9144-4; **Vol. II, Rings and Fields**, Marcel Dekker, Inc., New York, 1994, 531 pp., ISBN 0-8247-9159-2.

This outstanding textbook offers a *Comprehensive Self-Contained Presentation of all Major Topics in Abstract Algebra* and provides an in-depth treatment of the *Applications of Algebraic Techniques* and the *Relationship of Algebra* to other Disciplines such as *Number Theory*, *Combinatorics*, *Geometry*, *Topology*, *Differential Equations*, and *Markov Chains*.

Emphasizing both the conceptual and computational aspects of algebra to aid structural thinking as well as operational skills, the two volumes include through discussions of *Group Actions*, *Matrix Groups*, the *Correlation between Ring Theory*, *Number Theory and Algebraic Geometry*, *Galois Theory*, and the *Applicability of Linear Algebra*. Each chapter begins with an introduction that acts as a guide for the material to come, motivates and prepares students for new ideas with informal remarks, enlightening examples, and illustrations that facilitate understanding, furnishes abundant end-of-section problems of varying levels of difficulty to help to extend and apply their comprehension of newly learned material, allowing to use the text for a variety of different courses by maintaining essentially independent chapters. They supply an appendix that contains prerequisites from set theory and point-set topology and their proofs.

With over 230 drawings and numerous diagrams, tables, and display equations, the book is the perfect text for all upper-level undergraduate and graduate algebra and related courses, including *Abstract Algebra*, *Linear Algebra*, *Galois Theory*, *Commutative Algebra* and *Applications of Algebra*.

ABRAHAM BERMAN and ROBERT J. PLEMMONS, **Nonnegative Matrices in the Mathematical Sciences**. Science and Industry Advance with Mathematics, SIAM, Philadelphia, 1994, XX+340 pp., ISBN 0-89871-321-8.

Here is a valuable text and research tool for scientists and engineers who use of work with *Theory and Computation Associated with Practical Problems Relating to Markov Chains and Queuing Networks*, *Economic Analysis*, or *Mathematical Programming*. Originally published in 1979, this new edition adds material that updates and subject relative to developments from 1979 to 1993.

Theory and applications of nonnegative matrices are blended here, and extensive references are included in each area, from the theory of positive operators via Perron-Frobenius theory of non-

negative matrices and the theory of inverse positivity, to the widely used topic of M -matrices. On the way, semigroups of nonnegative matrices and symmetric nonnegative matrices are discussed. Later, applications of nonnegativity and M -matrices are given; for numerical analysis the example is convergence theory of iterative methods, for probability and statistics the examples are finite Markov chains and queueing network models, for mathematical economics the example is input-output models, and for mathematical programming the example is the linear complementarity problem.

C o n t e n t s. *Preface to the Classical Edition. Preface. Ch. 1, Matrices which Leave a Cone Invariant. Ch. 2, Nonnegative Matrices. Ch. 3, Semigroups of Nonnegative Matrices. Ch. 4, Symmetric Nonnegative Matrices. Ch. 5, Generalized Inverse-Positivity. Ch. 6, M-Matrices. Ch. 7, Iterative Methods for Linear Systems. Ch. 8, Finite Markov Chains. Ch. 9, Input-Output Analysis in Economics. Ch. 10, The Linear Complementarity Problem. Ch. 11, Supplement 1979-1993. References. Index.*

Exercises and biographical notes are included with each chapter.

As nonnegativity constraints arise very naturally throughout the physical world, engineers, applied mathematicians, and scientists who encounter nonnegativity or generalizations of nonnegativity in their work will benefit from topics covered here, connecting them to relevant theory. Researchers in one area, such as queueing theory, may find useful the techniques involving nonnegative matrices used by researchers in another area, say, mathematical programming.

P. NEITTAANMÄKI and D. TIBA, Optimal Control of Nonlinear Parabolic Systems. Theory, Algorithms, and Applications. Marcel Dekker, Inc. New York — Basel — Hong Kong, 1994, 399 pp., ISBN 0-8427-9081-2.

This book is devoted to *Optimal Control Problems Governed by Nonlinear Parabolic Systems Including Semilinear Equations, Variational Inequalities and Systems with Phase Transitions*. The authors aim to discuss a theoretical approach and then to present numerical methods (with convergence proofs or error estimates) necessary in computerizing optimal control processes.

This book is divided in 6 chapters; the readers interested only in nonlinear parabolic equations and their numerical solutions may confine themselves to the first three chapters. Chapters IV and V contain related control problems and for more advanced topics the Chapter VI is to be recommended.

In the beginning (Chs. I, II and III) there is a brief presentation of *Linear and Nonlinear Evolution Equations*, on the *Numerical Approximation via Finite Elements*, and *Convex Control Problems*. The authors also give a large number of examples which are discussed on several levels: theoretical analysis; related optimization problems; numerical problems. They stress on *Applications in Biochemistry, Chemistry, Biology, Multiphase Stefan-type Problems, Thermo-chemical Flow Phenomena, Continuous Casting of Steel, Crystal Growth and Sterilization of Canned Foods*.

The other Chs. (IV, V, VI) present certain advanced research topics as *Controllability of Free Boundaries* (the abstract results are applied to certain Stefan-type optimal control problems, proving controllability for the free boundary, and deriving special properties of the optimal control); *Error Analysis of Unbounded Nonlinear Parabolic Equations*; *Optimal Control Problems* (the distributed control for linear parabolic systems; the convex cost function; the optimal control and the state equations and finally the optimal control for the nonlinear parabolic systems); *State Constrained Systems and Optimal Control* (includes the presentation of optimality conditions and then a new approximation method for state constrained control problems).

This volume is intended for upper-level undergraduate and postgraduate students and researchers interested in nonlinear evolution systems, optimal control theory, numerical analysis optimization, and applications in sciences and engineering.

Gabriela Varvara

WILLIAM O. BRAY, P. S. MILOJEVIC and CASLAV V. STANOJEVIC (Eds.) **Fourier Analysis. Analytic and Geometric Aspects.** Lecture Notes in Pure and Applied Mathematics, 157, Marcel Dekker, Inc., New York – Basel – Hong Kong, 1994, 450 pp., ISBN 0-8247-9208-4.

This volume is based on the papers offered at the *Sixth Annual International Workshop in Analysis and Its Applications (IWAA)*, meeting held at the University of Maine, Orono, June 15-21, 1992. The principal theme was *Contemporary Aspects in Fourier Analysis*.

Fourier analysis has had and continues to have an unquestionable impact on virtually all other areas of mathematics and provides basic tools for the sciences. Abroad-based definition of Fourier analysis might take the form: analysis based on decomposition and synthesis formulas for complicated objects in terms of simpler objects intrinsically related to the underlying structure.

This volume contains complete expository and research papers focused on a variety of aspects of Fourier analysis, including *Topics in Classical Problems in the Theory of Trigonometric Transforms; Singular Integrals; Pseudodifferential Operators; Fourier Analysis on other Groups; Topics in Numerical Analysis and Wavelets*. As such, this volume addresses a wider mathematical audience than the previously proceedings, of interest and use not only to specialists in Fourier Analysis but also to graduate students and related fields.

The papers included in this volume are: *Ergodic and Mixing Properties of Radial Measures on the Heisenberg Group* (Mark A. G r a n o v s k y, Carlos B e r e n s t e i n, Der-Chen C h a n g); *H- and Weak H-Continuity of Calderon-Zygmund Type Operators* (Josefina A l v a r e z); *A New, Harder Proof that Continuous Functions with Schwarz Derivative 0 Are Lines* (Marshall A s h); *Integrability of Multiple Series* (Bruce A u b e r t i n, John J. F. F o u r n i e r); *Aspects of Harmonic Analysis on Real Hyperbolic Space* (William O. B r a y); *Trace Theorems via Wavelets on the Closed Set [0, 1]* (Anca D e l i u, Mong-Shu L e e); *Meta-Heisenberg Groups* (Gerald B. F o l l a n d); *Numerical Approximation of Singular Spectral Functions Arising from Fourier-Jacobi Problem on a Half Line with Continuous Spectra* (Charles T. F u l t o n, Steven P r u e s s); *Using Sums of Squares to Prove that Certain Entire Functions Have Only Real Zeros* (George G a s p e r); *An Application of Coxeter Groups to the Construction of Wavelet Bases in $\mathbb{R}$* (Jeffrey S. G e r o n i m o, Douglas P. H a r d i n, Peter R. M a s s o p u s t); *The Uniform Invertibility of Fourier Transforms of Compositions of Functions in $\mathbb{R}$ with Certain Quadratic Maps of $\mathbb{R}$* (Pete M. J a r v i s); *On Methods of Fourier Analysis in Multigrid Theory* (Erwin K r e y s z i g); *Orthogonal Wavelet Bases* (W. A. M a d y c h); *The Fourier Transform of Tempered Boehmians* (Piotr M i k u s i n s k i); *Approximation-Solvability of Nonlinear Equations and Applications* (P. S. M i l o j e v i c); *Homogeneous Cones and Homogeneous Integrals* (Tatiana O s t r o g r a d s k i); *Fourier Inversion in the Piecewise Smooth Category* (Mark A. P i n s k y); *Normed Linear Spaces of Trigonometric Transforms and Functions Analytic in the Unit Disk* (Caslav V. S t a n o j e v i c); *Fourier and Trigonometric Transforms with Complex Coefficients Regularly Varying in Mean* (Vera B. S t a n o j e v i c); *Sampling and the Multisensor Deconvolution Problem* (David F. W a l n u r).

Romulus Ștefan

GIOVANNI DORE (Ed.), **Differential Equations in Banach Spaces.** Marcel Dekker, Inc., New York, 1993, 270 pp., ISBN 0-8247-9067-7.

This valuable reference — based on the *Conference on Differential Equations* held in Bologna, Italy — provides up-to-date information on the most current research in parabolic and hyperbolic differential equations.

Presenting up-to-the minute *Methods and Results in Semigroup Theory and their Applications to Evolution Equations*, it focuses on *Abstract Parabolic and Hyperbolic Linear Differential Equations, Nonlinear Abstract Parabolic Equations, Holomorphic Semigroups, Volterra Operator*

Integral Equations, Euler-Bernoulli Equations and many other topics.

With important contributions from international experts, *Differential Equations in Banach Spaces* is an indispensable resource for research mathematicians in functional analysis, partial differential equations, operator theory, and control theory, and upper-level undergraduate and graduate students in these disciplines.

WILLIAM T. BRIGGS and VAN EMDEN HENSON, *The DFT. An Owner's Manual for the Discrete Fourier Transform*. Science and Industry Advance with Mathematics, SIAM, Philadelphia, 1995, 425 pp., ISBN 0-89871-342-0.

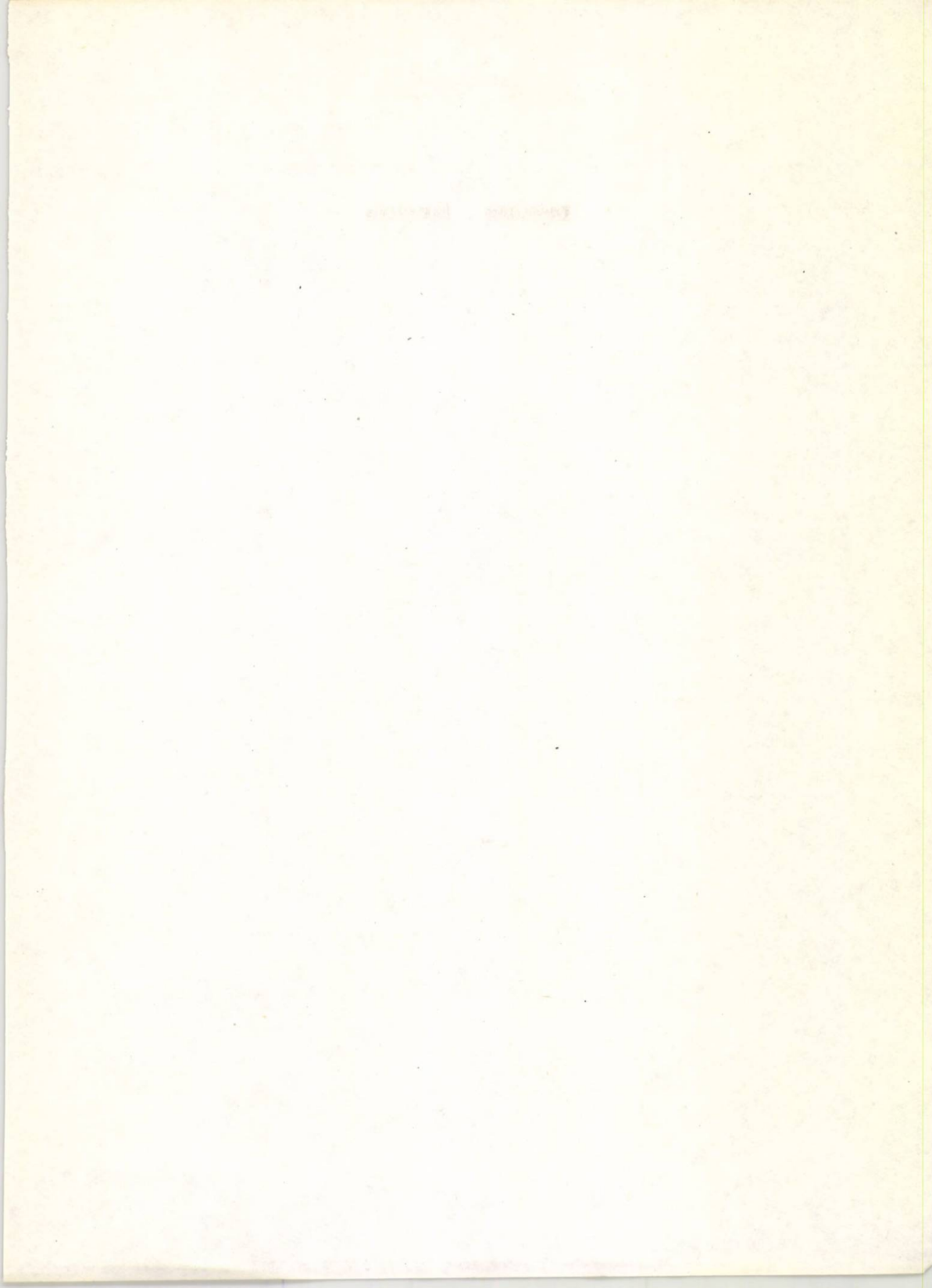
Just as a prism separates white light into its component bands of colored light, so the discrete Fourier transform (DFT) is used to separate a signal into its constituent frequencies. Just as a pair of sunglasses reduces the glare of white light, permitting only the softer green light to pass, so the DFT may be used to modify a signal to achieve a desired effect. In fact, by analyzing the component frequencies of a signal or any system, the DFT can be used in an astonishing variety of problems. Among the applications of the DFT are digital signal processing, oil and gas exploration, medical imaging, aircraft and spacecraft guidance, and the solution of differential equations of physics and engineering.

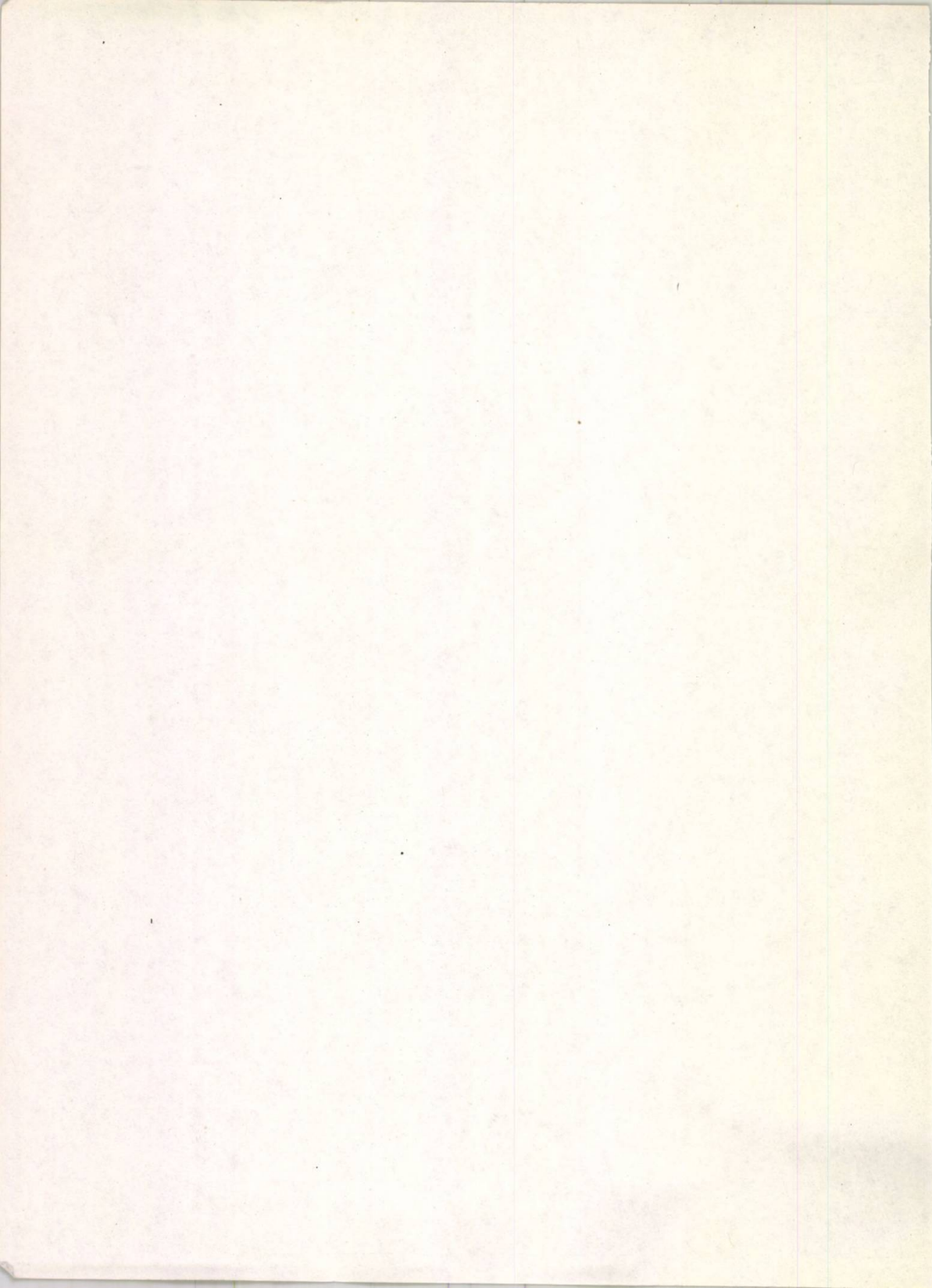
The book explores both the *Practical and Theoretical Aspects of the DFT*, one the most widely used tools in science, engineering, and computational mathematics. Designed to be accessible to an audience with diverse interests and mathematical backgrounds, the book is written in an informal style and is supported by many examples, figures and problems.

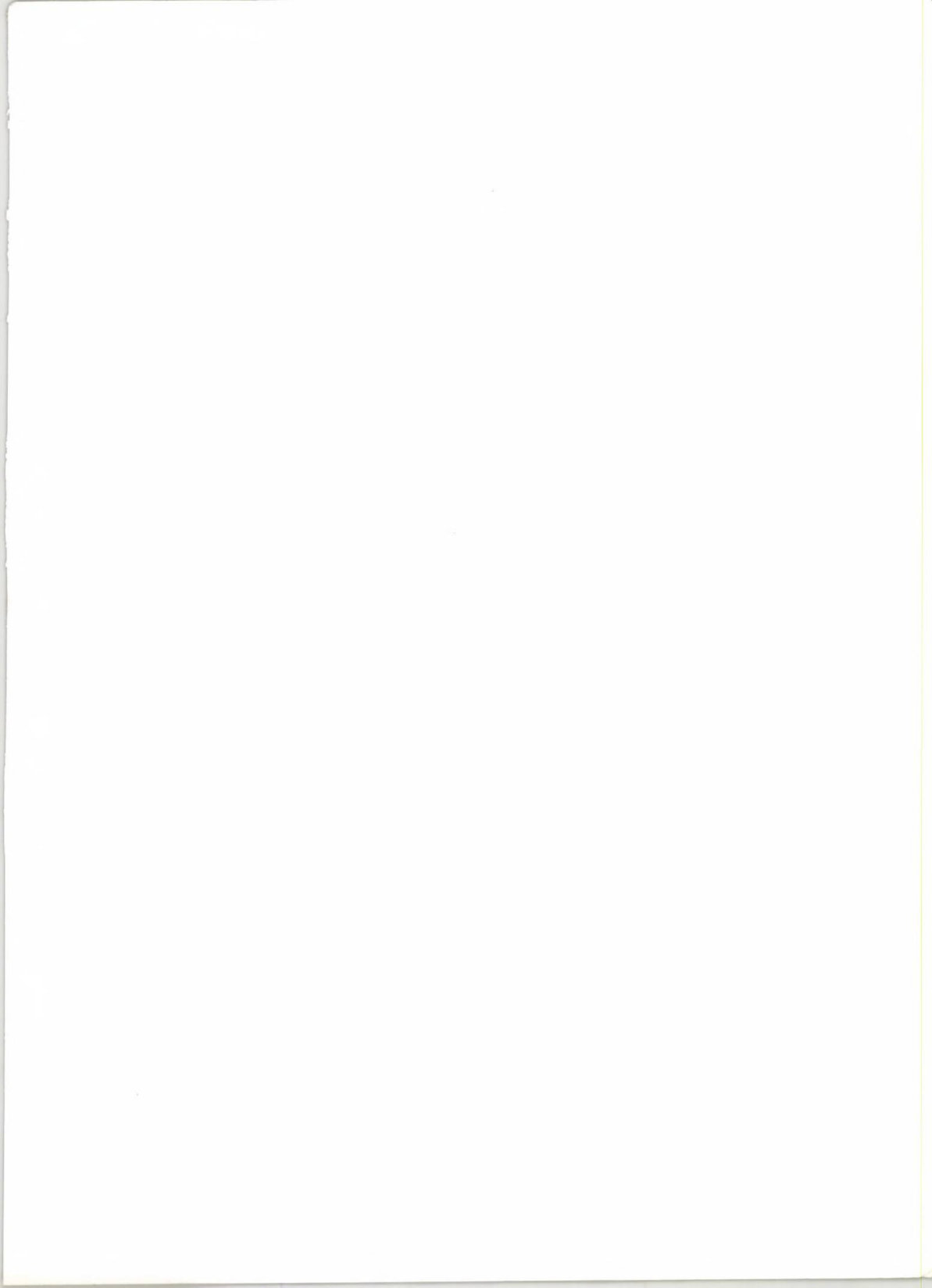
Conceived as an "owner's" manual, this comprehensive book covers such topics as the *History of the DFT, Derivations and Properties of the DFT, Comprehensive Error Analysis, Issues Concerning the Implementation of the DFT* in one and several dimensions, *Symmetric DFTs*, a sample of *DFT Applications*, and an *Overview of the FFT*.

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